Asset Pricing and Wealth Distribution with Heterogeneous Investment Returns

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Heterogeneous agent models

- Most heterogeneous agent models with incomplete markets are numerical.
- Analytical tractability (if possible) is nice because
  1. better grasp the structure of the model,
  2. more freedom to parameterize,
  3. allows us to estimate model.

Literature

Stochastic growth models (rate-of-return shocks)
: Krebs (2003a, 2003b, 2006), Angeletos & Calvet (2005, 2006), Angeletos (2007), Angeletos & Panousi (2011).

Bewley models (endowment shocks)
: Constantinides & Duffie (1996), Krueger & Lustig (2010), Krusell, Mukoyama, & Smith (2011)

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Investment returns

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- time: $t = 0, 1, 2, \ldots$.
- aggregate states: Markov chain on $\{1, 2, \ldots, S\}$.
- investment projects: $j \in \mathcal{J} = \{1, 2, \ldots, J\}$.
- Distribution of vector of returns on investment

$$\mathbf{R}_{t+1} = (R^1_{t+1}, \ldots, R^J_{t+1})$$

depends only on current aggregate state s_t.

end of time t	$\rightarrow$	beginning of time $t+1$
invest x^j	$\rightarrow$	get $R^j_{t+1} x^j$

Single agent problem

- $\Theta_s \subset \left\{\theta \in \mathbb{R}^J \,\middle|\, \theta^1 + \cdots + \theta^J = 1\right\}$: portfolio constraint (*e.g.*, leverage) in state s.
- $\theta_t \in \Theta_{s_t}$: portfolio at time t.
- $R_{t+1}(\theta_t) = \sum_{j=1}^J R_{t+1}^j \theta_t^j$: return on portfolio.
- Given initial wealth w_0, maximize Kreps-Porteus (1978), Epstein-Zin (1989) recursive CRRA/CEIS utility:

$$V(w,s) = \max_{\substack{c \geq 0 \\ \theta \in \Theta_s}} \left(c^{1-\sigma} + \beta \, \mathrm{E}\left[V(w', s')^{1-\gamma} \,\middle|\, s \right]^{\frac{1-\sigma}{1-\gamma}} \right)^{\frac{1}{1-\sigma}}$$

subject to $w' = R(\theta)(w - c)$,

where γ: relative risk aversion coefficient, $1/\sigma$: elasticity of intertemporal substitution.

- If $\sigma = \gamma$, usual additive CRRA utility.

Crucial condition

- Assume

$$(\forall s)\ \beta \left(\max_{\theta \in \Theta_s} \mathrm{E}\left[R(\theta)^{1-\gamma} \,\middle|\, s \right]^{\frac{1}{1-\gamma}} \right)^{1-\sigma} < 1. \qquad (\star)$$

- This condition guarantees the finiteness of recursive utility.
- Discount factor β cannot be too large,
  but $\beta < 1$ is not necessary (c.f. Kocherlakota 1990).

Optimal consumption/portfolio rule

Theorem

Suppose portfolio constraint Θ_s is nonempty, convex, compact and condition $(\star)$ holds. Then there exists a unique optimal consumption/portfolio rule. The optimal consumption rule is of the form $c(w,s) = a_s^{-\frac{1-\sigma}{\sigma(1-\gamma)}} w$, where $a_s > 0$. a_s and the optimal portfolio rule θ_s satisfy

$$a_s^{\frac{1-\sigma}{\sigma(1-\gamma)}} = 1 + \beta^{\frac{1}{\sigma}} \mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \,\middle|\, s\right]^{\frac{1-\sigma}{\sigma(1-\gamma)}},$$

$$\theta_s = \arg\max_{\theta \in \Theta_s} \mathrm{E}\left[a_{s'} R(\theta)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}.$$

Skip proof

Sketch of proof

1. Value function exists and of form $V(w,s) = a_s^{\frac{1}{1-\gamma}} w$ by condition $(\star)$.
2. Value function satisfies Bellman equation.
3. Substitute $V(w,s) = a_s^{\frac{1}{1-\gamma}} w$ into Bellman equation and construct a monotone mapping $\mathbf{a} \mapsto T\mathbf{a}$ on a compact set, hence exists a fixed point.
4. One of fixed points defines optimal consumption/portfolio rule.
5. (a_s, θ_s) can be obtained by iteration, hence constructive.

Comparative statics

Proposition

The saving rate (out of wealth) $1 - a_s^{-\frac{1-\sigma}{\sigma(1-\gamma)}}$ *is higher with more patience. If* $\sigma > 1$ (< 1), *the saving rate is higher (lower) with more risk.*

Proof.

Let $x_s = a_s^{\frac{1-\sigma}{\sigma(1-\gamma)}}$. Then $\mathbf{x} = (x_1, \dots, x_S)$ is a fixed point of

$$(T\mathbf{x})_s = 1 + \beta^{\frac{1}{\sigma}} \left(\max_{\theta \in \Theta_s} \mathrm{E}\left[x_{s'}^{\frac{\sigma(1-\gamma)}{1-\sigma}} R(\theta)^{1-\gamma} \,\middle|\, s \right]^{\frac{1}{1-\gamma}} \right)^{\frac{1-\sigma}{\sigma}}$$

Easy to show $T\mathbf{x} \le T'\mathbf{x}$ if $\beta < \beta'$ or $\mathbf{R}'$ is mean preserving spread of $\mathbf{R}$. Hence $\mathbf{x} = \lim T^n \mathbf{1} \le \lim (T')^n \mathbf{1} = \mathbf{x}'$. □

Public & private assets

- Arbitrary number of agents.
- Assume two kinds of assets, public and private, $\mathcal{J} = \mathcal{J}^{\mathrm{pub}} \cup \mathcal{J}^{\mathrm{priv}}$.
- $\mathbf{R}_{t+1} = (\mathbf{R}_{t+1}^{\mathrm{pub}}, \mathbf{R}_{t+1}^{\mathrm{priv}})$, where $\mathbf{R}_{t+1}^{\mathrm{priv}}$ is independent across agents conditional on current state s_t and returns on public assets $\mathbf{R}_{t+1}^{\mathrm{pub}}$.
- Some public assets (*e.g.*, risk-free asset) are in zero net supply: $\mathcal{J}^0 \subset \mathcal{J}^{\mathrm{pub}}$.

General equilibrium

Definition (Sequential equilibrium)

A sequential equilibrium consists of a consumption/portfolio rule $\{c_t, \theta_t\}$ (for each agent) and pricing kernels for public assets such that

1. $\{c_t, \theta_t\}$ are optimal subject to individual budget constraints, and
2. for $j \in \mathcal{J}^0$ the net supply of asset j is zero.

General equilibrium

Theorem (Existence & constrained efficiency)

Let $\Theta_s^0 = \left\{\theta \in \Theta_s \,\middle|\, \forall j \in \mathcal{J}^0, \theta^j = 0\right\}$ *be the portfolio constraint with holdings in assets in zero net supply restricted to be zero. Suppose that condition* $(\star)$ *holds with* Θ_s *replaced by* Θ_s^0*. Then a sequential equilibrium exists and can be constructed as follows.*

1. *the optimal consumption/portfolio rule is given by solving the single agent problem with* Θ_s *replaced by* Θ_s^0*, and*
2. *the pricing kernels are derived by the Euler equation.*

Furthermore, the equilibrium is constrained efficient.

Skip proof

Sketch of proof

1. Because of homothetic (CRRA/CEIS) preferences, everybody behaves in the same way, *i.e.*, same saving rate and same portfolio choice.
2. Hence the only way to clear assets in zero net supply is that nobody holds those assets.
3. Equilibrium is constrained efficient because solves single agent problem.

Wealth distribution

- Budget constraint $w_{t+1} = R_{t+1}(\theta_t)(w_t - c_t)$.
- Optimal consumption rule $c(w, s) = c_s w$, proportional to wealth.
- Hence wealth and consumption follow logarithmic random walks (with drift).
- If households infinitely lived, by CLT log wealth is

$$\log w_T = \log w_{\mathrm{ini}} + \sum_{t=1}^{T} X_{T-t} \sim \text{Gaussian},$$

  hence wealth growth $w_T / w_{\mathrm{ini}} \sim$ lognormal.
- Same for consumption.

Wealth distribution

- If households die with probability δ each period (discount factor β replaced by $\beta(1-\delta)$) and are reborn with some initial wealth, age distribution ν_δ is geometric with mean $1/\delta$.
- Hence log wealth is

$$\log w_T = \log w_{\text{ini}} + \sum_{t=1}^{\nu_\delta} X_{T-t} \sim \text{Laplace},$$

hence wealth growth $w_T / w_{\text{ini}} \sim$ double Pareto. Definition

- Same for consumption.

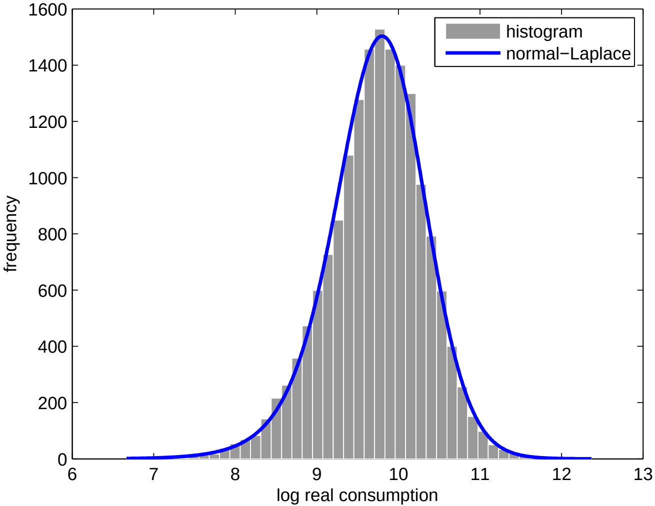

Figure: CEX (1985), taken from Toda & Walsh (2011).

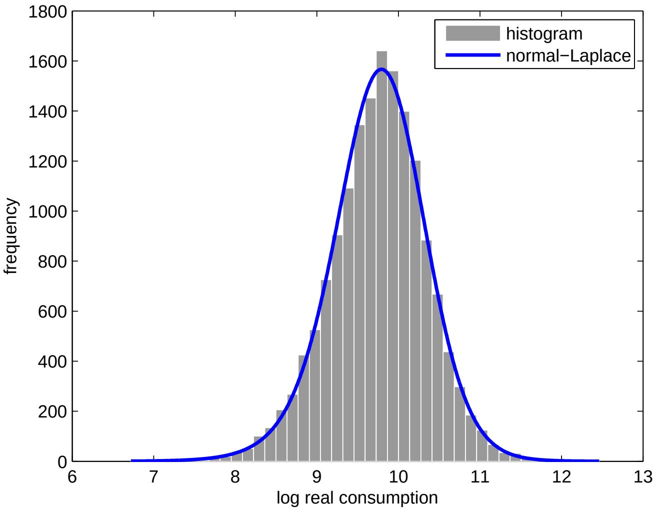

Figure: CEX (1990), taken from Toda & Walsh (2011).

Asset pricing

Proposition

If a traded asset pays $\{D_t\}_{t=0}^{\infty}$, *its price* $\{P_t\}_{t=0}^{\infty}$ *satisfies*

$$P_t = \frac{\mathrm{E}\left[a_{s_{t+1}} R(\theta_{s_t})^{-\gamma} (P_{t+1} + D_{t+1}) \,\middle|\, s_t\right]}{\mathrm{E}\left[a_{s_{t+1}} R(\theta_{s_t})^{1-\gamma} \,\middle|\, s_t\right]},$$

where a_s *is the coefficient of the value function and* θ_s *is the optimal portfolio. In particular, the risk-free rate* R_s *satisfies*

$$\frac{1}{R_s} = \frac{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]}.$$

Because $R(\theta_s)$ contains idiosyncratic shocks, market incompleteness has nontrivial effect on asset pricing.

Asset pricing

Proposition

The risk premium of the optimal portfolio $\mathrm{E}\left[R(\theta_s) \mid s\right] - R_s$ *is positive.*

Better be the case, but nontrivial.

Theorem (Covariance pricing)

Let R^j *be the return of asset* j, R_s *be the gross risk-free rate in state* s, *and* θ_s *be the optimal portfolio in state* s. *Then*

$$\mathrm{E}\left[R^j \mid s\right] - R_s = -\frac{\mathrm{Cov}\left[a_{s'} R(\theta_s)^{-\gamma}, R^j \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} \mid s\right]}.$$

Firms

- Preferences same as baseline model, but continuum of agents.
- Individuals rent physical and human capital to firms.
- Firm $j = 1, 2, \dots, J$ with neoclassical production function $F_{js}(K, H)$, where s: current state, K: physical capital, H: human capital.
- Firm's problem is static:

$$(\forall t) \quad \max_{K,H \geq 0} [F_{js_t}(K, H) - r_{jt}K - r_{0t}H],$$

where r_{jt}: rental rate of physical capital for firm j,
r_{0t}: rental rate of human capital.

Firms

- Individual capital ($j \geq 1$: physical, $j = 0$: human) evolves according to

$$k_{t+1}^j = z_{t+1}^j [(1 - \delta_t^j) k_t^j + x_t^j],$$

where

k_t^j	capital rented to firm j at beginning of period t,
δ_t^j	capital depreciation after production,
x_t^j	new investment to firm j ($j \geq 1$) or human capital ($j = 0$),
z_{t+1}^j	capital obsolescence or capital-augmenting technological change.

- Distribution of shocks $(z_{t+1}^j, \delta_{t+1}^j)_{j=0}^J$ depends only on current state s_t. Individual human capital shocks $(z_{t+1}^0, \delta_{t+1}^0)$ are conditionally independent across individuals.

General equilibrium

- Mathematically reduces to baseline model, but nontrivial.
- Let $\Theta_t \in \Delta^J$ be common portfolio choice of physical and human capital, and $\phi_t^j = H_t^j / \sum_j H_t^j \in \Delta^{J-1}$ be share of firm j human capital to aggregate human capital.
- By firm profit maximization, we have

$$r_{jt} = \frac{\partial}{\partial K} F_{js_t}(K_t^j, H_t^j) = \frac{\partial}{\partial K} F_{js_t}(z_t^j \Theta_{s_{t-1}}^j, \mathrm{E}_t[z_t^0]\Theta_{s_{t-1}}^0 \phi_t^j),$$
$$r_{0t} = \frac{\partial}{\partial H} F_{js_t}(K_t^j, H_t^j) = \frac{\partial}{\partial H} F_{js_t}(z_t^j \Theta_{s_{t-1}}^j, \mathrm{E}_t[z_t^0]\Theta_{s_{t-1}}^0 \phi_t^j).$$

- Can solve for $\left\{\phi_t^j\right\}$ and hence for $r_{jt} = r_j(s_t, \mathbf{z}_t | s_{t-1}, \Theta_{s_{t-1}})$.
- Define the return on individual portfolio θ_t by

$$R_{t+1}(\theta_t, \Theta_t) = \sum_j (1 + r_j(s_{t+1}, \mathbf{z}_{t+1} | s_t, \Theta_t) - \delta_{t+1}^j) z_{t+1}^j \theta_t^j.$$

General equilibrium

Theorem

Suppose condition similar to $(\star)$ holds. Then there exists an equilibrium with consumption rule $c(w,s) = a_s^{-\frac{1-\sigma}{\sigma(1-\gamma)}} w$ and portfolio rule θ_s, where

$$a_s^{\frac{1-\sigma}{\sigma(1-\gamma)}} = 1 + [\beta(1-\delta)]^{\frac{1}{\sigma}} \operatorname{E}\left[a_{s'} R(\theta_s, \Theta_s)^{1-\gamma} \,\middle|\, s\right]^{\frac{1-\sigma}{\sigma(1-\gamma)}},$$

$$\theta_s = \arg\max_{\theta \in \Delta^J} \operatorname{E}\left[a_{s'} R(\theta, \Theta_s)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}},$$

$$\theta_s = \Theta_s.$$

- $1-\delta$ because households die with probability $\delta \geq 0$.
- Similar to baseline model but agents maximize taking other agents' choice Θ_s as given.

Generic constrained inefficiency

Evolution of physical ($j \geq 1$) and human ($j = 0$) capital:

$$k^j_{t+1} = z^j_{t+1}[(1 - \delta^j_t)k^j_t + x^j_t].$$

Theorem

The equilibrium with firms is generically constrained inefficient. However, if the human capital-augmenting shock z^0_t is common across all consumers (i.e., the only idiosyncratic shock is in human capital depreciation), then the equilibrium is constrained efficient.

Intuition:

1. Households choose human capital investment before realization of shock.
2. Firms rent human capital after realization of shock.
3. Because of missing insurance market, over- or under-investment occurs.

Skip proof

Sketch of proof

1. The equilibrium is constrained efficient only if

$$\theta_s = \arg\max_{\theta \in \Delta^J} \mathrm{E}\left[a_{s'} R(\theta, \theta)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}.$$

2. However, by Theorem equilibrium satisfies

$$\theta_s = \arg\max_{\theta \in \Delta^J} \mathrm{E}\left[a_{s'} R(\theta, \Theta_s)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}$$

   with $\Theta_s = \theta_s$.

3. Since two maximizations are different, equilibrium is generically constrained inefficient.

Generalizations

Recursive CRRA/CEIS utility:

$$V(w,s) = \max_{\substack{c \geq 0 \\ \theta \in \Theta_s}} \left[c^{1-\sigma} + \beta \, \mathrm{E} \left[V(w', s')^{1-\gamma} \,\middle|\, s \right]^{\frac{1-\sigma}{1-\gamma}} \right]^{\frac{1}{1-\sigma}}$$

subject to $w' = R(\theta)(w - c)$.

Labor-leisure choice Replace c by $cv(l)$.

Multiple goods Replace c by $\left(\sum_l c_l^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$.

$\rightarrow$ Can apply to a New Keynesian model?

Bequest or utility generating asset (e.g., house)

Replace c by $\left(c^{1-\alpha} + bw^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$.

Conclusion

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2. Prove
   - existence of equilibrium (constructive),
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Double Pareto & Laplace distributions

- A nonnegative random variable X has a *double Pareto* distribution with mode M and power law exponents α, β if it has density

$$f_{\mathrm{dP}}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta}\frac{1}{M}\left(\frac{x}{M}\right)^{\beta-1}, & (0 \le x < M) \\ \frac{\alpha\beta}{\alpha+\beta}\frac{1}{M}\left(\frac{x}{M}\right)^{-\alpha-1}. & (x \ge M) \end{cases}$$

- If X is double Pareto, $\log X$ is *Laplace* with density

$$f_{\mathrm{L}}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta}\mathrm{e}^{-\beta|x-m|}, & (0 \le x < m) \\ \frac{\alpha\beta}{\alpha+\beta}\mathrm{e}^{-\alpha|x-m|}, & (x \ge m) \end{cases}$$

  where $m = \log M$ is the mode.
- If X: double Pareto, Y: lognormal, then XY: double Pareto-lognormal, $\log(XY)$: normal-Laplace.

Limit theorem

Theorem (Toda 2011)

Let $\{X_j\}$ *be independent but not identically distributed (i.n.i.d) with* $\mathrm{E}[X_j] = 0$ *and* $\mathrm{Var}[X_j] = \sigma_j^2$, *and* ν_p *be a geometric random variable independent of* X_j*'s with mean* $1/p$. *Suppose that*

1. $\lim_{n\to\infty} n^{-\alpha}\sigma_n^2 = 0$ *for some* $0 < \alpha < 1$ *and* $\sigma^2 := \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \sigma_j^2 > 0$ *exists, and*
2. *for all* $\epsilon > 0$ *we have*

$$\lim_{p\to 0} \sum_{j=1}^{\infty} (1-p)^{j-1} p\, \mathrm{E}\left[X_j^2 \left\{|X_j| \geq \epsilon p^{-\frac{1}{2}}\right\}\right] = 0.$$

Then, as $p \to 0$ *the geometric sum* $p^{\frac{1}{2}} \sum_{j=1}^{\nu_p} X_j$ *converges in distribution to a symmetric Laplace distribution with mean 0 and variance* σ^2. ▸ Go back