Necessity of Rational Asset Price Bubbles in Two-Sector Growth Economies

Tomohiro Hirano[1] Ryo Jinnai[2] Alexis Akira Toda[3]

[1]Royal Holloway, University of London

[2]Hitotsubashi University

[3]University of California San Diego

Asset price bubbles

- Casual inspection of modern economic history suggests presence of asset price bubbles
- Examples:
  - Japanese real estate bubble in late 1980s ▸ Image
  - U.S. dot-com bubble in late 1990s
  - U.S. housing bubble in mid 2000s
- Kindleberger (2000) documents 38 bubbly episodes in 1618–1998 period

Rational asset price bubbles

- Bubble: asset price (P) $>$ fundamental value (V)
  - $V =$ present value of dividends (D)
- Notoriously difficult to theoretically explain asset price bubbles in rational equilibrium models
- Santos and Woodford (1997) Bubble Impossibility Theorem: If present value of aggregate endowment $< \infty$, then $P = V$ whenever
  1. asset in positive net supply (e.g., land, stocks), or
  2. asset has finite maturity (e.g., bonds, options)
- Hence to generate bubbles in realistic settings, need financial frictions to prevent agents from capitalizing infinite present value

Shortcomings of existing rational bubble models

1. Pure bubble models ($D = 0$)
    - Literature almost exclusively focuses on $D = 0$
    - No empirical counterpart other than fiat money or cryptocurrency

Shortcomings of existing rational bubble models

1. Pure bubble models ($D = 0$)
   - Literature almost exclusively focuses on $D = 0$
   - No empirical counterpart other than fiat money or cryptocurrency
2. Equilibrium indeterminacy
   - $\exists$ fundamental equilibrium and continuum of bubbly equilibria
   - Hence model predictions non-robust

Shortcomings of existing rational bubble models

1. Pure bubble models ($D = 0$)
    - Literature almost exclusively focuses on $D = 0$
    - No empirical counterpart other than fiat money or cryptocurrency
2. Equilibrium indeterminacy
    - $\exists$ fundamental equilibrium and continuum of bubbly equilibria
    - Hence model predictions non-robust
3. Bubbles more likely under *tight* financial conditions
    - Stylized empirical facts (Kindleberger, 2000) suggest bubbles tend to be associated with *loose* financial conditions

Our contributions

1. Necessity of rational asset price bubbles
    - Present plausible economic models with bubbly equilibrium but without *any* fundamental equilibria
    - Hence bubbles no longer exotic; need to be embraced

Our contributions

1. Necessity of rational asset price bubbles
    - Present plausible economic models with bubbly equilibrium but without *any* fundamental equilibria
    - Hence bubbles no longer exotic; need to be embraced
2. Present two-sector endogenous growth model that circumvents all criticisms
    - Bubble attached to dividend-paying asset ("land")
    - Equilibrium determinate
    - As leverage constraint relaxed, inevitable phase transition from fundamental to bubbly regime

Related literature

- Pure bubble: Samuelson (1958), Bewley (1980), Tirole (1985), Kocherlakota (2009), Farhi and Tirole (2012), Aoki et al. (2014), Hirano and Yanagawa (2017), etc; see Miao (2014) and Ventura and Martin (2018) for reviews
- Impossibility results: Kocherlakota (1992), Santos and Woodford (1997)
- Macro-finance Brunnermeier and Sannikov (2014), etc.

Definition of bubbles

- Asset dividend $D_t \geq 0$, price $P_t \geq 0$ at $t = 0, 1, \dots$
- With Arrow-Debreu (date-0) price $q_t > 0$, no-arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}), \quad \text{so}$$
$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T \quad \text{by iteration}$$

Definition of bubbles

- Asset dividend $D_t \geq 0$, price $P_t \geq 0$ at $t = 0, 1, \dots$
- With Arrow-Debreu (date-0) price $q_t > 0$, no-arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}), \quad \text{so}$$

$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T \quad \text{by iteration}$$

- Letting $T \to \infty$, get

$$P_0 = \underbrace{\sum_{t=1}^{\infty} q_t D_t}_{=:V_0 = \text{fundamental value}} + \underbrace{\lim_{T\to\infty} q_T P_T}_{\text{transversality term}}$$

- If $\lim_{T\to\infty} q_T P_T = 0$, transversality condition holds and no bubble; if > 0, bubble

Plausible model with no fundamental equilibria

- Two agents with CRRA utility $\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$
- Aggregate endowment $(A+B)G^t$ with $A > B > 0$ and $G > 1$
- Asset pays constant dividend $D > 0$
- No shortsales allowed (Kocherlakota, 1992)
- For some $\eta > 0$, conjecture $R = G$, asset price & allocation

$$P_t = \underbrace{\frac{D}{G-1}}_{=V_0} + \underbrace{\eta G^t}_{\text{bubble}},$$

$$(c_{1t}, c_{2t}) = \begin{cases} ((B+\eta)G^t, (A-\eta)G^t) & \text{if } t\text{: even,} \\ ((A-\eta)G^t, (B+\eta)G^t) & \text{if } t\text{: odd} \end{cases}$$

- We shall specify individual endowments to support as equilibrium

Plausible model with no fundamental equilibria

- Let Agent 1 endowed with asset at $t = 0$,

$$e_{1t} = \begin{cases} BG^t - \frac{G}{G-1}D & \text{if } t\text{: even,} \\ AG^t + \frac{1}{G-1}D & \text{if } t\text{: odd,} \end{cases}$$

  flip even and odd endowments for Agent 2
- Agent 1 rich (poor) in odd (even) periods
- We have equilibrium if Euler equations satisfied:

$$\text{Poor:} \qquad \beta G \left(\frac{A-\eta}{B+\eta} G \right)^{-\gamma} \le 1,$$

$$\text{Rich:} \qquad \beta G \left(\frac{B+\eta}{A-\eta} G \right)^{-\gamma} = 1$$

Plausible model with no fundamental equilibria

Proposition

Let $\beta \in (0,1)$, $\gamma > 0$, $D > 0$. Take any $G > 1$ such that $\beta G^{1-\gamma} < 1$. Take any $A > B > 0$ such that

$$\eta := \frac{A(\beta G^{1-\gamma})^{1/\gamma} - B}{1 + (\beta G^{1-\gamma})^{1/\gamma}} > 0.$$

Then

1. *Consumption allocation $(c_t^r, c_t^p) = ((A - \eta)G^t, (B + \eta)G^t)$ and asset price $P_t = \frac{D}{G-1} + \eta G^t$ is bubbly equilibrium.*
2. *If $A > \beta^{-1/\gamma} GB$, then there exist no fundamental equilibria.*

Proof and intuition

- Easy to verify bubbly equilibrium by construction
- Nonexistence of fundamental equilibrium technical, but here is intuition:
    1. In equilibrium, Euler equation of rich implies

$$1 = \beta R_t (c^p_{t+1}/c^r_t)^{-\gamma} = \beta R_t \left(\frac{BG^{t+1} - \frac{1}{G-1}D + P_{t+1}}{AG^t + \frac{1}{G-1}D - P_t} \right)^{-\gamma}$$

    2. If $\exists$ fundamental equilibrium, P_t bounded, so

$$R_t = \frac{1}{\beta} \left(\frac{BG^{t+1} - \frac{1}{G-1}D + P_{t+1}}{AG^t + \frac{1}{G-1}D - P_t} \right)^{\gamma} \to \frac{1}{\beta}(BG/A)^{\gamma}$$

    3. If A large enough, $R_t < 1$, so $P_t = V_t = \infty$, contradiction

Necessity of rational asset price bubbles

- This stylized model shows bubbles are necessary for equilibrium existence in some models
- Key feature of these models: $G > G_d > R_f$, where
  - G: growth rate of economy
  - G_d: growth rate of dividends
  - R_f: risk-free rate in fundamental equilibrium (if exists)
- There exist no fundamental equilibria, for otherwise

$$V_0 = \sum_{t=1}^{\infty} D_0 (G_d/R_f)^t = \infty$$

- Rational bubble models are no longer exotic

Endogenous growth model overview

Key features of model:

- Infinite horizon: $t = 0, 1, \dots$
- Two sectors, production ("tech") & endowment ("land")
- Continuum of heterogeneous agents
- Incomplete markets
- Leverage constraint

Agents

- Continuum of agents with mass 1 indexed by $i \in I = [0,1]$
- Log utility (inessential; only for tractability)

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

- Rational expectations

Production ("tech" sector)

- AK-type technology; input $k_{it} \to$ output $y_{i,t+1} = z_{it} k_{it}$
- Productivity z_{it} IID across agents (can allow type dependence)
- z_{it} known at time t before investment

Assumption

Productivity z_{it} is IID *across agents with continuous cdf $F_t : [0, \infty) \to [0, 1]$ satisfying $F_t(1) < 1$ and $\int_0^\infty z \, \mathrm{d}F_t(z) < \infty$.*

Production ("tech" sector)

- AK-type technology; input $k_{it} \to$ output $y_{i,t+1} = z_{it} k_{it}$
- Productivity z_{it} IID across agents (can allow type dependence)
- z_{it} known at time t before investment

Assumption

Productivity z_{it} is IID across agents with continuous cdf $F_t : [0, \infty) \to [0, 1]$ satisfying $F_t(1) < 1$ and $\int_0^\infty z \, \mathrm{d}F_t(z) < \infty$.

- $F(1) < 1$ allows possibility of growth
- Finite mean $\int_0^\infty z \, \mathrm{d}F(z) < \infty$
- Continuity (no point mass) inessential but simplifies algebra

Land

- Unit supply of dividend-paying asset ("land")
  - e.g., housing, farmland, natural resources, traditional sector
- Land pays dividend $D_t \geq 0$, trades at price P_t

Assumption

Dividend satisfies $D_t > 0$ infinitely often.

Land

- Unit supply of dividend-paying asset ("land")
  - e.g., housing, farmland, natural resources, traditional sector
- Land pays dividend $D_t \geq 0$, trades at price P_t

Assumption

Dividend satisfies $D_t > 0$ infinitely often.

- Prevents land from becoming worthless
- Can show $P_t > 0$ for all t

Budget constraint

- Agents can trade risk-free bonds
- Exogenous supply B_t
  - Benchmark: $B_t = 0$ (closed economy)
  - Occasionally consider $B_t \neq 0$ (large open economy) to get closed-form solutions
- Gross risk-free rate R_t
- Budget constraint (suppressing i):

$$c_t + k_t + P_t x_t + b_t = \overbrace{\underbrace{z_{t-1} k_{t-1}}_{\text{production}} + \underbrace{(P_t + D_t) x_{t-1}}_{\text{land}} + \underbrace{R_{t-1} b_{t-1}}_{\text{bond}}}^{\text{income}},$$

where c_t: consumption; k_t: capital; x_t: land holdings, b_t: bond holdings

Leverage constraint

- Agents subject to leverage constraint

$$\underbrace{k_t}_{\text{investment}} \le \lambda_t(\underbrace{k_t + P_t x_t + b_t}_{\text{equity}}),$$

where $\lambda_t \ge 1$: leverage limit

- Note: equity $k_t + P_t x_t + b_t \ge k_t/\lambda_t \ge 0$ nonnegative
- Note: $P_t x_t + b_t \ge (1/\lambda_t - 1)k_t$, so joint shortsales constraint on land & bond (though assets can be shorted individually)

Rational expectations equilibrium

Definition

Given initial condition $\{(y_{i0}, x_{i,-1})\}_{i \in I}$ and bond supply $\{B_t\}_{t=0}^{\infty}$, rational expectations equilibrium consists of land prices $\{P_t\}_{t=0}^{\infty}$, interest rates $\{R_t\}_{t=0}^{\infty}$, and allocations $\{(c_{it}, k_{it}, x_{it}, b_{it})_{i \in I}\}_{t=0}^{\infty}$ such that

1. (Individual optimization) Agents maximize utility subject to budget & leverage constraints.
2. (Land market clearing) For all t, we have

$$\int_I x_{it} \, \mathrm{d}i = 1.$$

3. (Bond market clearing) For all t, we have

$$\int_I b_{it} \, \mathrm{d}i = B_t.$$

Individual optimization problem

- Define beginning-of-period wealth w_t by

$$c_t + k_t + P_t x_t + b_t = w_t$$
$$:= z_{t-1} k_{t-1} + (P_t + D_t) x_{t-1} + R_{t-1} b_{t-1}$$

- Define fraction of post-consumption wealth invested in production technology by $\theta_t = \frac{k_t}{w_t - c_t}$
- Land risk-free, so no-arbitrage implies $R_t = \frac{P_{t+1} + D_{t+1}}{P_t}$
- Hence budget constraint simplifies to

$$w_{t+1} = (\theta_t z_t + (1 - \theta_t) R_t)(w_t - c_t),$$

where leverage constraint implies $0 \leq \theta_t \leq \lambda_t$

Individual optimization problem

- Exploiting log utility, straightforward to solve for optimal consumption-investment rule
- Myopic consumption, leverage to max iff more productive than interest rate

Proposition (Optimal consumption and investment)

Suppose $\sup_t |\mathrm{E}[\log(R_t + \lambda_t \max\{0, z - R_t\})]| < \infty$*. Then optimal consumption-investment problem has essentially unique solution, which is given by*

$$c_t = (1-\beta) w_t,$$

$$\theta_t = \begin{cases} \lambda_t & \text{if } z_t > R_t, \\ \text{arbitrary} & \text{if } z_t = R_t, \\ 0 & \text{if } z_t < R_t. \end{cases}$$

Wealth dynamics

- Expected return is

$$\begin{aligned} \mathrm{E}_t[\theta_t z + (1-\theta_t)R_t] &= R_t + \lambda_t \int_0^\infty \max\{0, z - R_t\} \,\mathrm{d}F_t(z) \\ &=: R_t + \lambda_t \pi_t(R_t), \end{aligned}$$

  where $\pi_t(R) := \int_0^\infty \max\{0, z - R\} \,\mathrm{d}F_t(z)$ *risk premium* on unlevered capital
- Aggregating budget constraints (and using $c_t = (1-\beta)w_t$) yields aggregate wealth dynamics

$$W_{t+1} = \beta(\lambda_t \pi_t(R_t) + R_t)W_t$$

Market clearing

- Land & bond market clearing and optimal investment rule imply

$$\begin{aligned} P_t + B_t &= \int_I (P_t x_{it} + b_{it}) \, \mathrm{d}i \\ &= \underbrace{F_t(R_t) \times \beta W_t}_{\text{savings by unproductive}} + \underbrace{(1 - F_t(R_t)) \times \beta(1-\lambda_t) W_t}_{\text{borrowings by productive}} \\ &= \beta(\lambda_t F_t(R_t) + 1 - \lambda_t) W_t. \end{aligned}$$

- Define fraction of aggregate wealth flowing into asset market

$$\alpha_t := \beta(\lambda_t F_t(R_t) + 1 - \lambda_t)$$

- Putting all pieces together, get

Equilibrium dynamics

Proposition

Rational expectations equilibrium is characterized by:

$$\begin{aligned}
\alpha_t &= \beta(\lambda_t F_t(R_t) + 1 - \lambda_t), \\
P_t + B_t &= \alpha_t W_t, \\
W_0 &= \frac{Y_0 + D_0 - B_0}{1 - \alpha_0}, \\
W_{t+1} &= \beta(\lambda_t \pi_t(R_t) + R_t) W_t, \\
\beta(\lambda_t \pi_t(R_t) + R_t)\alpha_t &= R_{t-1}\alpha_{t-1} + \frac{B_t - R_{t-1}B_{t-1} - D_t}{W_{t-1}}.
\end{aligned}$$

- Last equation is just no-arbitrage $R_{t-1} = \frac{P_t + D_t}{P_{t-1}}$

Bubble Characterization Theorem

Theorem

Let $\{(P_t, R_t, B_t, (c_{it}, k_{it}, x_{it}, b_{it})_{i \in I})\}_{t=0}^{\infty}$ *be rational expectations equilibrium with associated aggregate wealth* $\{W_t\}_{t=0}^{\infty}$*. If*

$$\underbrace{\limsup_{t\to\infty} D_t < \infty,}_{\text{bounded dividend}} \quad \underbrace{\liminf_{t\to\infty} R_t > 1}_{\text{positive interest rate}}, \quad \underbrace{\liminf_{t\to\infty} \alpha_t > 0}_{\text{positive inflow to asset}},$$

and $\lim_{t\to\infty} B_t/W_t = 0$*, then following statements are true.*

1. *Fundamental value of land* V_t *is finite and* $\limsup_{t\to\infty} V_t < \infty$*.*
2. *If* $\limsup_{t\to\infty} W_t < \infty$*, then* $P_t = V_t$ *for all* t*, so land price equals fundamental value.*
3. *If* $\limsup_{t\to\infty} W_t = \infty$*, then* $P_t > V_t$ *for all* t*, so land price exceeds fundamental value (bubble).*

Bubble Characterization Theorem (less technical)

- $\exists$ bubbles $\iff$ aggregate wealth unbounded
- Intuition:
  1. Recall economy has two sectors (tech & land)
  2. In long run, economy either converges (tech $\approx$ land) or grows (tech $\gg$ land)
  3. If converge, W finite $\implies$ P finite ($\because P \leq W$) $\implies$ transversality condition holds ▸ ? $\implies$ no bubble
  4. If growth, $W \to \infty$ $\implies$ $P \to \infty$ ($\because$ land held by unproductive agents, but must be able to finance productive investment) $\implies$ $P > V$ eventually $\implies$ $P > V$ always ($\because$ backward induction)

Bubble Characterization Theorem (less technical)

- $\exists$ bubbles $\iff$ aggregate wealth unbounded
- Intuition:
  1. Recall economy has two sectors (tech & land)
  2. In long run, economy either converges (tech $\approx$ land) or grows (tech $\gg$ land)
  3. If converge, W finite $\implies$ P finite ($\because P \leq W$) $\implies$ transversality condition holds ▸ ? $\implies$ no bubble
  4. If growth, $W \to \infty$ $\implies$ $P \to \infty$ ($\because$ land held by unproductive agents, but must be able to finance productive investment) $\implies$ $P > V$ eventually $\implies$ $P > V$ always ($\because$ backward induction)
- Key features to deliver necessity of bubbles
  1. Incomplete markets (for $W \to \infty$ while consistent with optimization)
  2. Two sectors (to decouple aggregate economy from land sector)

Long run equilibria

- Bubble Characterization Theorem assumes existence of equilibrium with $W_t \to W$ or $W_t \to \infty$
- Analysis vacuous unless construct such equilibria
- Consider long run setting where variables constant (or constant growth)
  - Assume $D_t = D > 0$, $\lambda_t = \lambda \geq 1$ (constant)
  - Assume $R_t \to R > 1$, $W_t / W_{t-1} \to G > 0$ (constant growth), $B_t / W_t \to 0$ (long run bond clearing)
- From wealth dynamics ▸ ?, get $G = \beta(\lambda\pi(R) + R)$

Trend stationary equilibria

Definition

Suppose $F_t = F$, $D_t = D > 0$, and $\lambda_t = \lambda \geq 1$. We say that rational expectations equilibrium $\{(P_t, R_t, B_t, (c_{it}, k_{it}, x_{it}, b_{it})_{i \in I})\}_{t=0}^{\infty}$ with associated aggregate wealth $\{W_t\}_{t=0}^{\infty}$ is *trend stationary equilibrium* if following conditions hold.

1. (Constant interest rate) $R_t = R > 0$ for all t.
2. (Constant growth rate) $W_t / W_{t-1} = G > 0$ for all t.
3. (Long run bond market clearing) $\lim_{t \to \infty} B_t / W_t = 0$.

Existence and local determinacy of equilibrium

- Necessary and sufficient condition for existence of fundamental trend stationary equilibrium ▸ Details
- Necessary and sufficient condition for existence of (unique) bubbly trend stationary equilibrium ▸ Details
- Local determinacy of bubbly long run equilibrium (not necessarily trend stationary) ▸ Details
    - Local determinacy is in sharp contrast to literature, where equilibrium is indeterminate

Financial conditions and emergence of bubbles

Theorem (Inevitability of bubbles with lax leverage)

Suppose $\Pr(z > 1/\beta) > 0$*. Then there exists leverage threshold* $\bar{\lambda}$ *such that all long run equilibria are bubbly if* $\lambda \geq \bar{\lambda}$*.*

Financial conditions and emergence of bubbles

Theorem (Inevitability of bubbles with lax leverage)

Suppose $\Pr(z > 1/\beta) > 0$*. Then there exists leverage threshold* $\bar{\lambda}$ *such that all long run equilibria are bubbly if* $\lambda \geq \bar{\lambda}$*.*

Proof.

- In fundamental equilibrium, $1/\beta = \lambda\pi(R) + R$
- Equilibrium forces $1 < R < 1/\beta$
- Hence $1/\beta > \lambda\pi(1/\beta) + 1$, so λ cannot be too high □

Phase transition to bubble economy

- If $G = 1$ (no growth), then equilibrium condition is

$$G = \beta(\lambda\pi(R) + R) = 1 \iff \phi_f(R) := \frac{\beta}{1-\beta R}\pi(R) = \frac{1}{\lambda}$$

- If $G > 1$ (growth), letting $t \to \infty$ in no-arbitrage condition yields

$$G = \beta(\lambda\pi(R) + R) = R \iff \phi_b(R) := \frac{\beta}{1-\beta}\frac{\pi(R)}{R} = \frac{1}{\lambda}$$

- Can show $\phi_f(R) \geq \phi_b(R)$

Phase transition to bubble economy

Proposition

If $\Pr(z > 1/\beta) > 0$ *and* $\mathrm{E}[z \mid z \geq 1] > 1/\beta$*, then*

1. ϕ_f *is strictly increasing for* $R \in [1, 1/\beta)$ *and* $\phi_f(1/\beta) = \infty$*,*
2. ϕ_b *is strictly decreasing whenever* $\phi_b > 0$ *and* $\phi_b(\infty) = 0$*.*

Letting $\mathcal{R}_f, \mathcal{R}_b$ *be sets of fundamental & bubbly equilibrium interest rates,*

1. *if* $1/\lambda > \frac{\beta}{1-\beta}\pi(1)$*, then* $\mathcal{R}_f$ *is singleton and* $\mathcal{R}_b = \emptyset$*, and*
2. *if* $1/\lambda < \frac{\beta}{1-\beta}\pi(1)$*, then* $\mathcal{R}_b$ *is singleton and* $\mathcal{R}_f = \emptyset$*.*

Phase transition to bubble economy

Proposition

If $\Pr(z > 1/\beta) > 0$ *and* $\mathrm{E}[z \mid z \geq 1] > 1/\beta$, *then*

1. ϕ_f *is strictly increasing for* $R \in [1, 1/\beta)$ *and* $\phi_f(1/\beta) = \infty$,
2. ϕ_b *is strictly decreasing whenever* $\phi_b > 0$ *and* $\phi_b(\infty) = 0$.

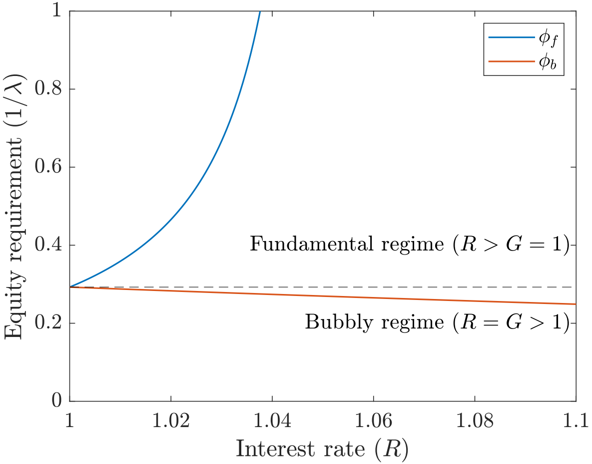
Letting $\mathcal{R}_f, \mathcal{R}_b$ *be sets of fundamental & bubbly equilibrium interest rates,*

1. *if* $1/\lambda > \frac{\beta}{1-\beta}\pi(1)$, *then* $\mathcal{R}_f$ *is singleton and* $\mathcal{R}_b = \emptyset$, *and*
2. *if* $1/\lambda < \frac{\beta}{1-\beta}\pi(1)$, *then* $\mathcal{R}_b$ *is singleton and* $\mathcal{R}_f = \emptyset$.

- In plain language: as leverage λ relaxed, economy inevitably transitions from fundamental regime with $R > G = 1$ to bubbly regime with $R = G > 1$

Phase transition to bubble economy with lax leverage

- Example: $1 - F(z) = \kappa e^{-z/\bar{z}}$ with $\kappa = 0.02$, $\bar{z} = 1.5$, $\beta = 0.95$ Skip



Wealth dynamics

- Using budget constraint and optimal rules, individual wealth evolves according to

$$w_{i,t+1} = \beta(\lambda \max\{0, z_{it} - R\} + R)w_{it}$$

- Logarithmic random walk, so no stationary distribution if agents infinitely lived
- Assume Yaari (1965)-type perpetual youth model with survival probability $\upsilon < 1$ each period
- Analysis identical if wealth of dead agents redistributed to newborn agents (inessential)

Wealth distribution

- If $G = 1$, letting $s_t = w_{t+1}/W_{t+1}$ be relative wealth, get

$$s_t = \begin{cases} (1 + (1 - \beta R)g(z_t))s_{t-1} & \text{with probability } \upsilon, \\ 1, & \text{with probability } 1 - \upsilon, \end{cases}$$

where $g(z) := \frac{\max\{0, z - R\}}{\pi(R)} - 1$

- If $G = R > 1$, get

$$s_t = \begin{cases} (1 + (1 - \beta)g(z_t))s_{t-1} & \text{with probability } \upsilon, \\ 1, & \text{with probability } 1 - \upsilon \end{cases}$$

- Both Markov multiplicative process with reset, hence stationary distribution has Pareto upper tail by Beare and Toda (2022)

Pareto exponent

Theorem

Suppose $\Pr(z > R) > 0$. *Then*

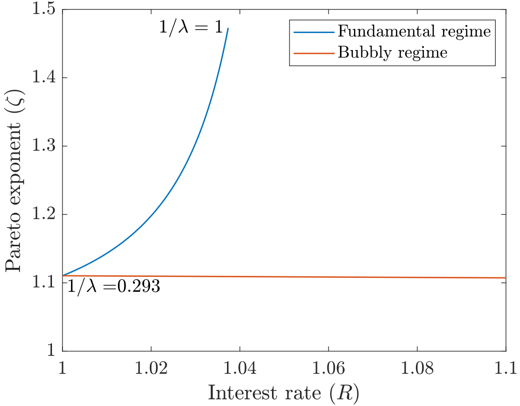
1. *Stationary distribution has Pareto upper tail with exponent* $\zeta > 1$: $\lim_{s \to \infty} s^\zeta \Pr(s_t > s) \in (0, \infty)$ *exists.*
2. *Pareto exponent* ζ *is uniquely determined by*

$$1 = \rho(\zeta) := \begin{cases} \upsilon \, \mathrm{E}[(1 + (1 - \beta R) g(z))^\zeta] & \text{if } G = 1, \\ \upsilon \, \mathrm{E}[(1 + (1 - \beta) g(z))^\zeta] & \text{if } G > 1. \end{cases}$$

3. *Letting* $\zeta_f(R), \zeta_b(R) > 1$ *be Pareto exponents in fundamental and bubbly regime given equilibrium interest rate* $R > 1$, *we have* $\zeta_f(R) > \zeta_b(R)$.

Example

- Same example, set $\upsilon = 0.99$
- In fundamental regime, leverage $\lambda \uparrow \implies R \downarrow, \zeta \downarrow$
- In bubbly regime, $\lambda \uparrow \implies R \uparrow, \zeta \rightarrow$
- Bubble allows unproductive agents to "catch up"



Extensions

- Model is stylized but many assumptions can be relaxed
  1. Could be homothetic utility instead of log
  2. Could have Markov productivity shock instead of IID
  3. Productivity distribution could have atoms (discontinuous cdf)
  4. Could have dividend growth
- See paper for details

Concluding remarks

- Necessity of rational asset price bubbles for equilibrium existence
    - $\exists$ plausible economic models with bubbly equilibrium but without *any* fundamental equilibria
- Two-sector endogenous growth model circumvents all criticisms to pure bubble models
    - Bubble attached to dividend-paying asset ("land")
    - Equilibrium determinate
    - Inevitable phase transition from fundamental to bubbly regime with lax leverage

Concluding remarks

- Necessity of rational asset price bubbles for equilibrium existence
    - $\exists$ plausible economic models with bubbly equilibrium but without *any* fundamental equilibria
- Two-sector endogenous growth model circumvents all criticisms to pure bubble models
    - Bubble attached to dividend-paying asset ("land")
    - Equilibrium determinate
    - Inevitable phase transition from fundamental to bubbly regime with lax leverage
- Door to bubble economy opened, many interesting questions
    - Equilibrium selection in pure bubble by dividend injection
    - Rational housing bubbles (rent = endogenous dividend)

References

Aoki, K., T. Nakajima, and K. Nikolov (2014): "Safe Asset Shortages and Asset Price Bubbles," *Journal of Mathematical Economics*, 53, 164–174.

Beare, B. K. and A. A. Toda (2022): "Determination of Pareto Exponents in Economic Models Driven by Markov Multiplicative Processes," *Econometrica*, 90, 1811–1833.

Bewley, T. (1980): "The Optimum Quantity of Money," in *Models of Monetary Economies*, ed. by J. H. Kareken and N. Wallace, Federal Reserve Bank of Minneapolis, 169–210.

Brunnermeier, M. K. and Y. Sannikov (2014): "A Macroeconomic Model with a Financial Sector," *American Economic Review*, 104, 379–421.

Farhi, E. and J. Tirole (2012): "Bubbly Liquidity," *Review of Economic Studies*, 79, 678–706.

Hirano, T. and N. Yanagawa (2017): "Asset Bubbles, Endogenous Growth, and Financial Frictions," *Review of Economic Studies*, 84, 406–443.

Kindleberger, C. P. (2000): *Manias, Panics, and Crashes*, New York: John Wiley & Sons, fourth ed.

Kocherlakota, N. R. (1992): "Bubbles and Constraints on Debt Accumulation," *Journal of Economic Theory*, 57, 245–256.

——— (2009): "Bursting Bubbles: Consequences and Cures," Unpublished manuscript.

Miao, J. (2014): "Introduction to Economic Theory of Bubbles," *Journal of Mathematical Economics*, 53, 130–136.

Samuelson, P. A. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy*, 66, 467–482.

Santos, M. S. and M. Woodford (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19–57.

Tirole, J. (1985): "Asset Bubbles and Overlapping Generations," *Econometrica*, 53, 1499–1528.

Ventura, J. and A. Martin (2018): "The Macroeconomics of Rational Bubbles: A User's Guide," *Annual Review of Economics*, 10, 505–539.

Yaari, M. E. (1965): "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," *Review of Economic Studies*, 32, 137–150.

Existence of fundamental equilibrium

- Define set of candidate interest rate by

$$\mathcal{R}_f := \{R > 1 : 1 - F(R) < 1/\lambda, \lambda\pi(R) + R = 1/\beta\}.$$

Theorem (Existence, $G \leq 1$)

A fundamental trend stationary equilibrium exists if and only if $\mathcal{R}_f \neq \emptyset$. Under this condition, variables must satisfy

$$G = 1, \qquad R \in \mathcal{R}_f, \qquad B_t = 0,$$
$$W_t = \frac{D}{(R-1)\alpha}, \qquad P_t = \frac{D}{R-1}, \qquad Y_0 = \frac{1-R\alpha}{(R-1)\alpha}D.$$

Return

Existence of bubbly equilibrium

- Define set of candidate interest rate by

$$\mathcal{R}_b := \left\{ R > 1 : 1 - F(R) < 1/\lambda, \lambda \frac{\pi(R)}{R} = 1/\beta - 1 \right\}.$$

Theorem (Existence and uniqueness, $G > 1$)

Let $\underline{R} = \max\left\{1, F^{-1}(1/\lambda - 1)\right\}$. Then a bubbly trend stationary equilibrium with $1 - F(R) \neq 1/\lambda$ exists if and only if $\lambda\pi(\underline{R})/\underline{R} > 1/\beta - 1$. Under this condition, equilibrium is unique and variables must satisfy

$$G = R, \qquad R \in \mathcal{R}_b, \qquad B_t = -\frac{D}{R-1},$$

$$W_t = R^t \frac{(R-1)Y_0 + RD}{(1-\alpha)(R-1)}, \quad P_t = \alpha W_t + \frac{D}{R-1}.$$

Local determinacy of bubbly equilibrium

Theorem (Local determinacy)

Let everything be as above, $B_t = B$ *(constant), and suppose that* F *is differentiable. Then for large enough initial aggregate endowment* $Y_0 > 0$*, there exists a unique bubbly long run equilibrium.*

Local determinacy of bubbly equilibrium

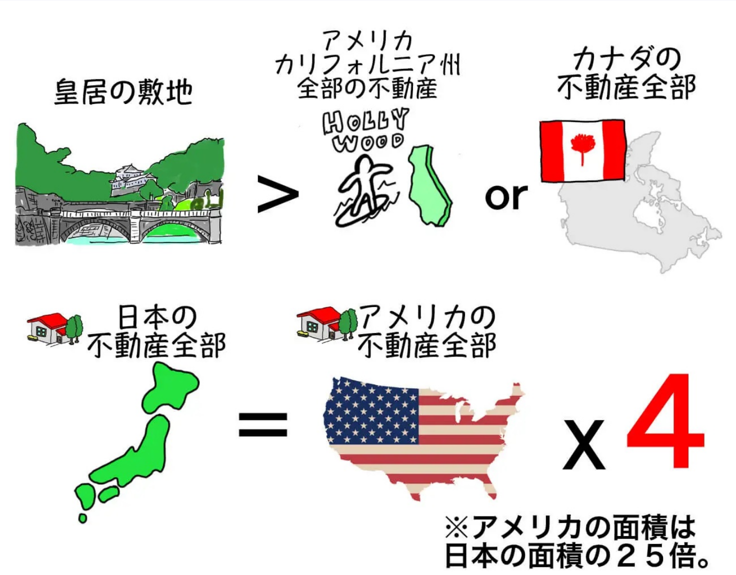
Theorem (Local determinacy)

Let everything be as above, $B_t = B$ (constant), and suppose that F is differentiable. Then for large enough initial aggregate endowment $Y_0 > 0$, there exists a unique bubbly long run equilibrium.

- Previous theorem established existence & uniqueness of bubbly *trend stationary* equilibria
- Hence multiplicity in equilibria that are not trend stationary not obvious
- This theorem rules out multiplicity (in sharp contrast to literature with equilibrium indeterminacy)

Return

Figure: Japanese bubble in late 1980s


皇居の敷地
>
アメリカ
カリフォルニア州
全部の不動産
HOLLY
WOOD
or
カナダの
不動産全部
日本の
不動産全部
=
アメリカの
不動産全部
x 4
※アメリカの面積は
日本の面積の２５倍。

Return