Bubble Necessity Theorem

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Rational asset price bubbles

- Bubble: asset price (P) $>$ fundamental value (V)
  - $V =$ present value of dividends (D)

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- Bubble: asset price (P) $>$ fundamental value (V)
  - $V =$ present value of dividends (D)
- Bubbles are often considered special or fragile:

> *Our main results are concerned with nonexistence of asset pricing bubbles in those economies. These results imply that the conditions under which bubbles are possible—including some well-known examples of monetary equilibria—are relatively fragile.*
>
> *—abstract of Santos and Woodford (1997)*

Pure bubble models

- It is well known that bubbles are possible
  - Samuelson (1958): bubbles in OLG model
  - Bewley (1980): bubbles in infinite-horizon model
  - See Hirano and Toda (2024) for recent review
- Existing literature focuses on *pure bubbles*
  - asset pays no dividends ($D = 0$)
  - hence intrinsically worthless ($V = 0$)

Possibility versus necessity of bubbles

- In pure bubble models, $V = 0$ is always equilibrium (fundamental equilibrium)
- In many models, there also exist continuum of bubbly equilibria
- Hence bubbles are possible but not necessary (inevitable)

Contribution

- We prove Bubble Necessity Theorem in plausible general class of economic models
  - plain vanilla general equilibrium model
  - there exist equilibria
  - in all equilibria, $P > V$
- Bubble necessity condition: $R < G_d < G$, where
  - G: economic growth rate
  - G_d: dividend growth rate
  - R: (counterfactual) autarky interest rate
- Modern macro-finance theory seems to presuppose $P = V$; we challenge this view and claim $P > V$ is norm under unbalanced growth

Related literature

- Sufficient condition for bubbles Okuno and Zilcha (1983), Aiyagari and Peled (1991): if autarky inefficient, then $\exists$ bubbly equilibrium
- Necessary condition for bubbles Kocherlakota (1992), Santos and Woodford (1997): if $\exists$ bubble, then PV of aggregate endowment $= \infty$
- Nonexistence of fundamental equilibria Wilson (1981)
    - Our marginal contribution: making it a general and formal theorem

Definition of bubbles

- Asset dividend $D_t \geq 0$, price $P_t \geq 0$ at $t = 0, 1, \dots$
- With Arrow-Debreu (date-0) price $q_t > 0$, no-arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}), \quad \text{so}$$

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- Letting $T \to \infty$, get

$$P_0 = \underbrace{\sum_{t=1}^{\infty} q_t D_t}_{\text{fundamental value } V_0} + \underbrace{\lim_{T\to\infty} q_T P_T}_{\text{bubble component}}$$

- If $\lim_{T\to\infty} q_T P_T = 0$, transversality condition holds and no bubble; if > 0, bubble

Bubble Characterization Lemma (Montrucchio, 2004)

Lemma

If $P_t > 0$ for all t, asset price exhibits bubble if and only if

$$\sum_{t=1}^{\infty} \frac{D_t}{P_t} < \infty$$

Bubble Characterization Lemma (Montrucchio, 2004)

Lemma

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$$\sum_{t=1}^{\infty} \frac{D_t}{P_t} < \infty$$

- Hence bubble if and only if sum of dividend yields finite
- Since $\sum_{t=1}^{\infty} 1/t = \infty$ but $\sum_{t=1}^{\infty} 1/t^{\alpha} < \infty$ for $\alpha > 1$, $\exists$ bubble if price-dividend ratio P_t/D_t grows faster than linearly

Proof

- From no-arbitrage condition $q_{t-1}P_{t-1} = q_t(P_t + D_t)$, get

$$\frac{q_{t-1}P_{t-1}}{q_t P_t} = 1 + \frac{D_t}{P_t}$$

- Taking product from $t = 1$ to $t = T$, get

$$\frac{q_0 P_0}{q_T P_T} = \prod_{t=1}^{T} \left(1 + \frac{D_t}{P_t}\right)$$

- Expanding terms and using $1 + x \le \mathrm{e}^x$, get

$$1 + \sum_{t=1}^{T} \frac{D_t}{P_t} \le \frac{q_0 P_0}{q_T P_T} \le \exp\left(\sum_{t=1}^{T} \frac{D_t}{P_t}\right)$$

- Let $T \to \infty$ and use definition of TVC

Two-sector growth economy with land

- Two-period OLG model, utility $(1-\beta)\log y + \beta \log z$
- Two sectors with production functions

$$F_{1t}(H,X) = G_1^t H,$$
$$F_{2t}(H,X) = G_2^t H^\alpha X^{1-\alpha},$$

where H: labor/human capital, X: land

  - Sector 1 labor-intensive (service, finance, information, etc.)
  - Sector 2 land-intensive (agriculture, construction, etc.)

- Assume $G_1 > G_2$, so productivity growth higher in Sector 1

Equilibrium

- Equilibrium is sequence

$$\{(P_t, r_t, w_t, x_t, y_t, z_t, H_{1t}, H_{2t})\}_{t=0}^{\infty},$$

where P_t: land price, r_t: land rent, w_t: wage, x_t: land holdings, (y_t, z_t): young and old consumption, (H_{1t}, H_{2t}): labor input

- Utility/profit maximization, market clearing (good, land, labor)
- Profit maximization:

$$\alpha G_2^t H_{2t}^{\alpha-1} = w_t = G_1^t \iff H_{2t} = \alpha^{\frac{1}{1-\alpha}} (G_2/G_1)^{\frac{t}{1-\alpha}}$$

- Rent: $X = 1$ implies

$$r_t = (1-\alpha) G_2^t H_{2t}^{\alpha} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} G_2^t (G_2/G_1)^{\frac{\alpha t}{1-\alpha}}$$

Necessity of land bubble

- Young consumption $y_t = (1-\beta)w_t = (1-\beta)G_1^t$
- In equilibrium, young must buy land: $x_t = 1$
- Hence land price

$$P_t = P_t x_t = w_t - y_t = \beta G_1^t$$

- Dividend yield

$$\frac{r_t}{P_t} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} G_2^t (G_2/G_1)^{\frac{\alpha t}{1-\alpha}}}{\beta G_1^t} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\beta}(G_2/G_1)^{\frac{t}{1-\alpha}},$$

summable because $G_1 > G_2$, so by Bubble Characterization Lemma ▸?, land bubble is inevitable

GDP share of agriculture decreases with income

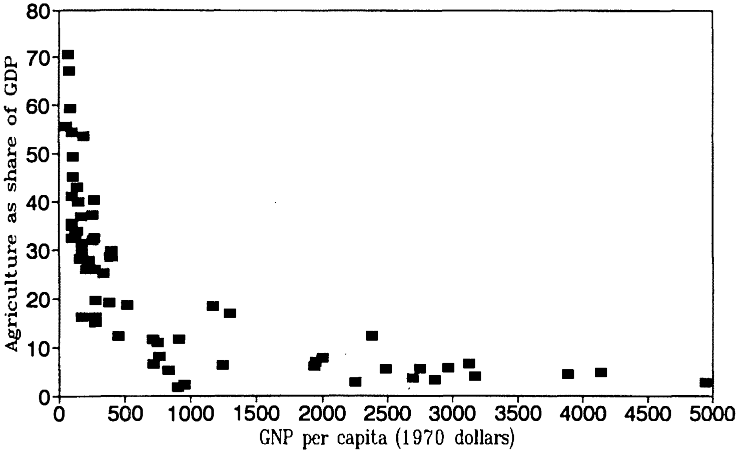

Figure Echevarria (1997, Figure 2)

Employment share of agriculture decreases over time

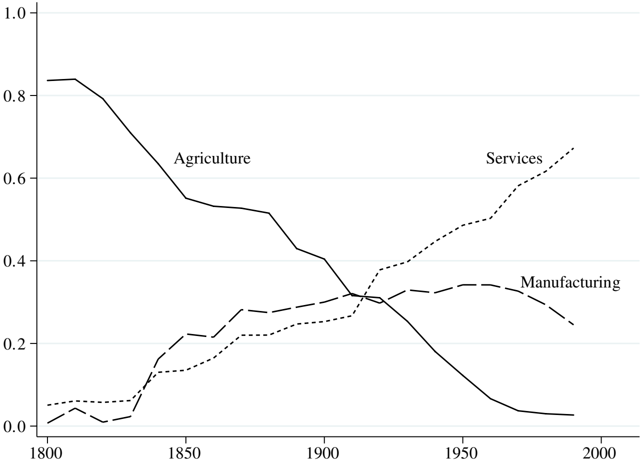

FIGURE 20.1 The share of U.S. employment in agriculture, manufacturing, and services, 1800–2000.

Figure Acemoglu (2009, Figure 20-1)

Innovation and stock market bubble

- Two-period OLG model, utility $(1-\beta)\log y + \beta \log z$
- Neoclassical aggregate production function $F(K, L)$, where K: capital, L: labor
- Capital K_t and labor L_t exogenous (inessential)
- Rent: $r_t = F_K(K_t, L_t)$
- Wage: $w_t = F_L(K_t, L_t)$

Dividend yield

- As before, price of stock (claim to rent) is

$$P_t = \beta w_t L_t = \beta F_L(K_t, L_t) L_t$$

- Dividend equals aggregate rents:

$$D_t = r_t K_t = F_K(K_t, L_t) K_t$$

- Hence dividend yield is

$$\frac{D_t}{P_t} = \frac{1}{\beta} \frac{F_K(K_t, L_t) K_t}{F_L(K_t, L_t) L_t}$$

Necessity of stock market bubble

- Suppose K_t, L_t grow at rates G_K, G_L
- Suppose F exhibits constant elasticity of substitution (CES), so

$$F(K, L) = \left(\alpha K^{1-1/\sigma} + (1-\alpha)L^{1-1/\sigma}\right)^{\frac{1}{1-1/\sigma}}$$

- Then dividend yield is

$$\frac{D_t}{P_t} = \frac{\alpha}{\beta(1-\alpha)}\left((G_K/G_L)^t (K_0/L_0)\right)^{1-1/\sigma}$$

- Hence if $G_K > G_L$ (so technological progress faster than labor productivity growth) and $\sigma < 1$ (consistent with empirical evidence), then D_t/P_t summable and stock market bubble

Model

- Two period OLG model
- Utility of generation t is $U_t(y_t, z_{t+1})$
- Time t endowments of young and old are (a_t, b_t)
- Long-lived asset pays dividend $D_t \geq 0$
- Budget constraints are

$$\begin{aligned} \text{Young:} \quad & y_t + P_t x_t = a_t, \\ \text{Old:} \quad & z_{t+1} = b_{t+1} + (P_{t+1} + D_{t+1}) x_t, \end{aligned}$$

where P_t: asset price, x_t: asset holdings of young

Equilibrium

- Equilibrium notion is competitive equilibrium with sequential trading

Definition

A competitive equilibrium consists of a sequence of prices $\{P_t\}_{t=0}^{\infty}$ and allocations $\{(x_t, y_t, z_t)\}_{t=0}^{\infty}$ satisfying the following conditions:

1. (Individual optimization) The initial old consume $z_0 = b_0 + P_0 + D_0$; for all t, the young maximize utility $U_t(y_t, z_{t+1})$ subject to the budget constraints
2. (Commodity market clearing) $y_t + z_t = a_t + b_t + D_t$ for all t
3. (Asset market clearing) $x_t = 1$ for all t

Bubbly and asymptotically bubbly equilibria

Definition

An equilibrium is *fundamental* (*bubbly*) if $P_0 = V_0$ ($P_0 > V_0$).

- Definition of bubbly equilibria obvious

Bubbly and asymptotically bubbly equilibria

Definition

An equilibrium is *fundamental* (*bubbly*) if $P_0 = V_0$ ($P_0 > V_0$).

- Definition of bubbly equilibria obvious
- However, want to rule out bubbly equilibria that are asymptotically bubbleless

Definition (Asymptotically bubbly equilibria)

Let $\{P_t\}_{t=0}^{\infty}$ be equilibrium asset prices. The asset is *asymptotically relevant (irrelevant)* if

$$\liminf_{t\to\infty} \frac{P_t}{a_t} > 0 \quad (= 0).$$

A bubbly equilibrium is *asymptotically bubbly (bubbleless)* if the asset is asymptotically relevant (irrelevant).

Assumptions

Assumption (A1)

For all t, the utility function $U_t : \mathbb{R}^2_+ \to [-\infty, \infty)$ is continuous, quasi-concave, and continuously differentiable on $\mathbb{R}^2_{++}$ with positive partial derivatives.

- Standard assumption
- Convenient to define marginal rate of substitution

$$M_t(y, z) := \frac{(U_t)_z(y, z)}{(U_t)_y(y, z)} > 0$$

Existence of equilibrium

Theorem (Existence)

If A1 holds, an equilibrium exists. The asset prices satisfy $0 \le P_t \le a_t$ *and*

$$P_t = \min\left\{M_t(y_t, z_{t+1})(P_{t+1} + D_{t+1}), a_t\right\},$$

where $(y_t, z_{t+1}) = (a_t - P_t, b_{t+1} + P_{t+1} + D_{t+1})$.

- Note that budget constraint $y_t + P_t x_t = a_t$ and market clearing $x_t = 1$ forces $y_t = a_t - P_t$
- Proof is by truncation & Tychonoff's theorem

Assumptions

- We now aim to prove necessity of bubbles under additional assumptions

Assumption (A2)

The endowments $\{(a_t, b_t)\}_{t=0}^{\infty}$ *satisfy*

$$\lim_{t\to\infty} \frac{a_{t+1}}{a_t} =: G \in (0, \infty),$$
$$\lim_{t\to\infty} \frac{b_t}{a_t} =: w \in [0, \infty).$$

- Asymptotically constant income growth and ratio

Assumptions

- Define scaled forward rate function $f_t : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_t(y,z) := \frac{1}{M_t(a_t y, a_t z)} = \frac{(U_t)_y(a_t y, a_t z)}{(U_t)_z(a_t y, a_t z)}$$

- We impose following uniform convergence condition on f_t

Assumption (A3)

There exists a continuous function $f : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $f_t \to f$ uniformly on compact sets, that is, for any nonempty compact set $K \subset \mathbb{R}_{++} \times \mathbb{R}_+$, we have

$$\lim_{t\to\infty} \sup_{(y,z)\in K} |f_t(y,z) - f(y,z)| = 0.$$

CRRA example

- Suppose U exhibits constant relative risk aversion (CRRA), so $U_t(y,z) = u(y) + \beta u(z)$ with

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } 0 < \gamma \neq 1, \\ \log c & \text{if } \gamma = 1 \end{cases}$$

- Here $\beta > 0$ is discount factor and $\gamma > 0$ is relative risk aversion coefficient
- Then

$$f_t(y,z) = f(y,z) = \frac{1}{\beta}(z/y)^{\gamma},$$

so A3 obviously holds

Bubble Necessity Theorem

Theorem (Bubble Necessity Theorem)

If A1–A3 hold and

$$f(1, Gw) < G_d := \limsup_{t \to \infty} D_t^{1/t} < G,$$

then all equilibria are asymptotically bubbly.

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If A1–A3 hold and

$$f(1, Gw) < G_d := \limsup_{t\to\infty} D_t^{1/t} < G,$$

then all equilibria are asymptotically bubbly.

- This is the bubble necessity condition $R < G_d < G$

Intuition

- If fundamental (or asymptotically bubbleless) equilibrium exists, because $G_d < G$, asset becomes asymptotically irrelevant ($P_t \sim G_d^t \ll G^t$)
- Then equilibrium autarky in long run, and interest rate converges to

$$R_t = \frac{1}{M_t(y_t, z_{t+1})} \to f(1, Gw) < G_d$$

- But then fundamental value of asset infinite, so $P_t \geq V_t = \infty$, contradiction
- Hence all equilibria must be asymptotically bubbly

Proof

- Proof is technical and highly nontrivial (as we want to prove that *all* equilibria are asymptotically bubbly)
- Here we mention steps
  1. Let $d_t = D_t/a_t$; show d_t is summable and hence $d_t \to 0$
  2. Let $p_t = P_t/a_t$; show p_{t+1}/p_t is bounded above by universal constant (use Euler equation)
  3. Show that if p_t sufficiently small, then $p_{t+1}/p_t < 1$ (use $f(1, Gw) < G$)
  4. If $\exists$ asymptotically bubbleless equilibrium, then (by definition) p_t gets arbitrarily close to 0, and hence must converge to 0 by previous step
  5. Derive a contradiction (use $f(1, Gw) < G_d$)

Example, linear utility

- Suppose agents have linear utility $U(y,z) = y + \beta z$
- Then $f(y,z) = 1/\beta$
- Bubble necessity condition is $1/\beta < G_d < G$
- Wilson (1981)'s example uses $G = 1$, $G_d = 1/2$, and $\beta = 3$

Example, CRRA utility

- Suppose agents have CRRA utility

$$U(y,z)=\frac{y^{1-\gamma}}{1-\gamma}+\beta\frac{z^{1-\gamma}}{1-\gamma}$$

- Suppose endowments (aG^t, bG^t), dividend $D>0$
- Then $f(y,z)=\frac{1}{\beta}(z/y)^\gamma$
- Bubble necessity condition is

$$\frac{1}{\beta}(bG/a)^\gamma<1<G \iff a>\beta^{-1/\gamma}Gb,$$

so bubbles are inevitable whenever young (saver) are sufficiently rich

Robustness

- Paper discusses extensions to Bewley-type infinite-horizon models with
  - idiosyncratic investment shocks (model closest to Kiyotaki (1998))
  - idiosyncratic preference shocks (model closest to Chien and Wen (2022))

Concluding remarks

- Bubbles have generally been considered special or fragile
- Existing literature studies possibility of bubbles ("bubbles *can* arise")
- We proved the necessity of bubbles ("bubbles *must* arise") in some well-behaved economies
- It may open up new directions for research
    - Leverage and bubbles (Hirano et al., 2022)
    - Housing bubbles (Hirano and Toda, 2023a)
    - Unbalanced growth and bubbles (Hirano and Toda, 2023b)

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