Note on Bubbles Attached to Real Assets

Tomohiro Hirano[1] Alexis Akira Toda[2]

[1]Royal Holloway, University of London

[2]Emory University

Emory University
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Asset price bubble

- Asset price bubble: situation in which "asset prices do not reflect fundamentals" (Stiglitz, 1990)
  - Asset price (P) $>$ fundamental value (V)
  - Fundamental value (V) $=$ present value of dividends (D)
- Many approaches to asset price bubble
  - rational bubble, heterogeneous beliefs, asymmetric information, econometric, empirical, etc.
- We are mostly interested in so-called "rational bubble" models
  - $P > V$ as equilibrium outcome with rational agents with common belief and information

Difficulty of bubbles attached to real assets

- By real assets, we mean $D > 0$ (e.g., land, housing, stocks)
- Fundamental difficulty in generating rational bubbles in real assets
  - Santos and Woodford (1997, Theorem 3.3, Corollary 3.4): bubbles impossible if dividends non-negligible relative to endowments
  - See Hirano and Toda (2024a, §3.4) for simple illustration
- As a result, existing rational bubble literature almost exclusively focus on "pure bubble" models with $D = 0$
  - Classic papers: Samuelson (1958), Bewley (1980), Tirole (1985), Kocherlakota (1992), etc.

Criticisms to pure bubble models

1. $D = 0$ is unrealistic
   - Examples other than fiat money or cryptocurrency?
2. Equilibrium indeterminacy (existence of continuum of bubbly equilibria) (Gale, 1973; Hirano and Toda, 2024b)
3. With $D = 0$, price-dividend ratio P/D undefined, so cannot connect to econometric literature on bubble detection (Phillips and Shi, 2018)

This paper

- We wrote this paper for a book chapter edited by Manuel Santos
- We discuss recent development of rational bubbles attached to real assets
    1. Necessity of bubbles (Hirano and Toda, 2025a): Under some conditions, bubbles are inevitable
    2. Bubbles attached to real assets are nonstationary, but can be analyzed using local stable manifold theorem
    3. Close connection between bubbles and economic development
- As application, present simple model with stock and land price bubble

Definition of bubbles

- Asset dividend $D_t \geq 0$, price $P_t \geq 0$ at $t = 0, 1, \dots$
- With Arrow-Debreu (date-0) price $q_t > 0$, no-arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}), \quad \text{so}$$

$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T \quad \text{by iteration}$$

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- Letting $T \to \infty$, get

$$P_0 = \underbrace{\sum_{t=1}^{\infty} q_t D_t}_{=:V_0=\text{fundamental value}} + \underbrace{\lim_{T\to\infty} q_T P_T}_{\text{bubble component}}$$

- If $\lim_{T\to\infty} q_T P_T = 0$, no bubble; if > 0, bubble
- $\lim_{T\to\infty} q_T P_T$ captures speculative motive (buy now to sell later)

Bubble Characterization Lemma

Lemma (Montrucchio, 2004)

If $P_t > 0$ for all t, asset price exhibits bubble if and only if

$$\sum_{t=1}^{\infty} \frac{D_t}{P_t} < \infty$$

- Hence bubble if and only if sum of dividend yields finite
- Except pure bubble models ($D_t \equiv 0$), bubbles are fundamentally nonstationary phenomena: price must grow faster than dividend

Proof

- By no-arbitrage,

$$q_{t-1}P_{t-1} = q_t(P_t + D_t) \iff \frac{q_{t-1}P_{t-1}}{q_t P_t} = 1 + \frac{D_t}{P_t}$$

- Taking product from $t = 1$ to $t = T$, get

$$\frac{q_0 P_0}{q_T P_T} = \prod_{t=1}^{T} \left(1 + \frac{D_t}{P_t}\right)$$

- Expanding terms and using $1 + x \le \mathrm{e}^x$, we obtain

$$1 + \sum_{t=1}^{T} \frac{D_t}{P_t} \le \frac{q_0 P_0}{q_T P_T} \le \exp\left(\sum_{t=1}^{T} \frac{D_t}{P_t}\right)$$

- Let $T \to \infty$ and use definition of bubble

OLG model with log utility

- Two-period overlapping generations (OLG) model
- Utility of generation t

$$U(y_t, z_{t+1}) = (1-\beta)\log y_t + \beta \log z_{t+1},$$

  where (y_t, z_{t+1}): consumption when young and old
- Endowment: $a_t > 0$ when young, 0 when old
- Long-lived asset in unit supply, dividend $D_t > 0$ (exogenous)

OLG model with log utility

- Letting x_t asset holdings, budget constraint:

$$\begin{aligned} \text{Young:} \quad & y_t + P_t x_t = a_t, \\ \text{Old:} \quad & z_{t+1} = (P_{t+1} + D_{t+1}) x_t \end{aligned}$$

- Equilibrium: $\{(P_t, x_t, y_t, z_t)\}_{t=0}^{\infty}$ such that agents maximize utility and markets clear
  - Commodity clearing: $y_t + z_t = a_t + D_t$
  - Asset clearing: $x_t = 1$

OLG model with log utility

- Cobb-Douglas utility implies $y_t = (1-\beta)a_t$
- Hence asset price $P_t = P_t x_t = a_t - y_t = \beta a_t$
- Hence $\exists$ unique equilibrium, with dividend yield $D_t/P_t = D_t/\beta a_t$

Proposition

There exists a unique equilibrium, and the asset price exhibits a bubble if and only if $\sum_{t=1}^{\infty} D_t/a_t < \infty$.

OLG model with linear utility

- Example of Wilson (1981, §7)
- Linear utility $U(y_t, z_{t+1}) = y_t + \beta z_{t+1}$
- Endowments $(a_t, b_t) = (aG^t, bG^t)$
- Dividends $D_t = DG_d^t$

Proposition

If $1/\beta < G_d < G$, then the unique equilibrium asset price is $P_t = aG^t$, and there is a bubble.

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- Intuition: if equilibrium interior, interest rate $R = 1/\beta < G_d$, so PV of dividends $= \infty$, contradiction
- Hence $y_t = 0$, asset price $P_t = aG^t$, and dividend yield $D_t/P_t = (D/a)(G_d/G)^t$ summable

Other examples

- Tirole (1985)'s OLG model with capital and labor: Bosi, Ha-Huy, Le Van, Pham, and Pham (2018), Hirano and Toda (2025a, §V), Pham and Toda (2025)
- Bewley (1980)'s two-agent economy: Le Van and Pham (2016), Bosi, Le Van, and Pham (2022)
- See paper for details

Necessity of bubbles

- We have seen a few examples of bubbly equilibria with real assets, but generality unclear
- Here we briefly explain Bubble Necessity Theorem of Hirano and Toda (2025a)
- OLG model with utility $U(y_t, z_{t+1})$
- Endowment of young and old (a_t, b_t), dividend $D_t > 0$
- Budget constraint

$$\begin{aligned} \text{Young:} \quad & y_t + P_t x_t = a_t, \\ \text{Old:} \quad & z_{t+1} = b_{t+1} + (P_{t+1} + D_{t+1}) x_t \end{aligned}$$

Equilibrium

- Equilibrium defined as before
- $x_t = 1$ forces equilibrium allocation

$$(y_t, z_{t+1}) = (a_t - P_t, b_{t+1} + P_{t+1} + D_{t+1})$$

- First-order condition

$$U_y(y_t, z_{t+1})P_t = U_z(y_t, z_{t+1})(P_{t+1} + D_{t+1}),$$

- Interest rate

$$R_t := \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{U_y}{U_z}(y_t, z_{t+1})$$

- Arrow-Debreu price $q_0 = 1$ and $q_t = 1/\prod_{s=0}^{t-1} R_s$, fundamental value $V_0 = \sum_{t=1}^{\infty} q_t D_t$

Bubble Necessity Theorem

- Define long-run dividend growth rate $G_d := \limsup_{t\to\infty} D_t^{1/t}$
- Assume $(a_t, b_t) \to (a, b)$
- Define autarky interest rate $R := (U_y/U_z)(a, b)$

Theorem

If $R < G_d < 1$, then all equilibria are bubbly with $\liminf_{t\to\infty} P_t > 0$.

- Intuition: if equilibrium fundamental, because $G_d < 1$, asset price small and $(y_t, z_{t+1}) \to (a, b)$
- Then $R < G_d$ implies fundamental value infinite, contradiction

Other necessity results

- Two-period OLG model is for simplicity, but not necessary
- Hirano and Toda (2025a) discuss other necessity results
    - §V.A: bubble necessity in OLG model with capital and labor
    - §V.B: bubble necessity in infinite-horizon model with log utility and idiosyncratic productivity shocks
    - §V.C: bubble necessity in infinite-horizon model with quasi-linear utility and idiosyncratic liquidity shocks
- Hirano and Toda (2025b) prove necessity of land overvaluation with aggregate risk

Long-run behavior

- We have seen that bubbles attached to real assets are necessarily nonstationary (because D_t/P_t is summable)
- How do we study such models? Local stable manifold theorem (local linearization)!
- For illustration, suppose
  - Utility $U(y, z)$ homothetic (homogeneous of degree 1)
  - Endowments $(a_t, b_t) = (aG^t, bG^t)$
  - Dividends $D_t = DG_d^t$ with $G_d < G$

Fundamental equilibrium

- Assume first equilibrium is fundamental, so $P_t \sim G_d^t$
- Define detrended price $p_t := G_d^{-t} P_t$
- First-order condition is

$$U_y p_t = G_d U_z(p_{t+1} + D),$$

evaluated at

$$(y, z) = (aG^t - p_t G_d^t, bG^{t+1} + (p_{t+1} + D)G_d^{t+1})$$

- By homotheticity, can also evaluate at

$$(y, z) = (a - p_t(G_d/G)^t, Gb + G_d(p_{t+1} + D)(G_d/G)^t)$$

Definition of detrended system

- Define auxiliary variable $\xi_t = (\xi_{1t}, \xi_{2t}) \in \mathbb{R}^2_{++}$ by $\xi_{1t} = p_t = P_t/G_d^t$, $\xi_{2t} = (G_d/G)^t$
- Then we can write system as $\Phi(\xi_t, \xi_{t+1}) = 0$, where $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi_1(\xi, \eta) = G_d(\eta_1 + D)U_z - \xi_1 U_y,$$
$$\Phi_2(\xi, \eta) = \eta_2 - (G_d/G)\xi_2,$$

  where $(\xi, \eta) = (\xi_1, \xi_2, \eta_1, \eta_2)$ and $\Phi = (\Phi_1, \Phi_2)$
- Partial derivatives evaluated at

$$(y, z) = (a - \xi_1\xi_2, Gb + G_d(\eta_1 + D)\xi_2).$$

Local behavior around steady state

- Idea of analysis is as follows
  1. Compute steady state by solving $\Phi(\xi^*, \xi^*) = 0$
  2. Around steady state, apply implicit function theorem to solve as $\xi_{t+1} = \phi(\xi_t)$ with $\xi^* = \phi(\xi^*)$
  3. Applying local stable manifold theorem, local behavior of ξ_t same as linear difference equation $\xi_{t+1} - \xi^* = A(\xi_t - \xi^*)$ for some matrix A
  4. If number of unstable eigenvalues of A ($|\lambda| > 1$: unstable, $|\lambda| < 1$: stable) equals number of endogenous initial condition (e.g., P_0), then equilibrium locally determinate
- Too technical to cover details: see paper and references therein
- After tedious calculations, we can show:

Fundamental long-run equilibria

Proposition

The following statements are true.

1. *There exists a unique* $w = w_f^*$ *satisfying* $(U_y/U_z)(1, Gw) = G_d$.
2. *There exists a steady state* ξ^* *of* Φ *if and only if* $b/a > w_f^*$. *Under this condition, there exists a unique path* $\{\xi_t^*\}_{t=0}^{\infty}$ *converging to* ξ^*.
3. *The corresponding equilibrium asset price has order of magnitude*

$$P_t = \xi_{1t}^* G_d^t \sim \frac{G_d U_z}{U_y - G_d U_z} D G_d^t,$$

and there is no asset price bubble.

Bubbly long-run equilibria

Proposition

The following statements are true.

1. *There exists a unique* $w = w_b^* > w_f^*$ *satisfying* $(U_y/U_z)(1, Gw) = G$.
2. *There exists a bubbly steady state* $\xi^* > 0$ *of* Φ *if and only if* $b/a < w_b^*$. *Under this condition, there exists a path* $\{\xi_t^*\}_{t=0}^{\infty}$ *converging to* ξ^* *if* $d \neq 0, -n$. *The path is unique if* $d > 0$.
3. *The corresponding equilibrium asset price has order of magnitude*

$$P_t = \xi_{1t}^* G^t \sim \frac{w_b^* a - b}{1 + w_b^*} G^t,$$

and there is an asset price bubble.

Complete analysis

- By detrending as $p_t = G^{-t} P_t$, fundamental steady state $\xi^* = 0$ always exists
- Additional calculations yield following complete analysis

Theorem

Let $w = b/a$ be the old-to-young income ratio and define $w_f^* < w_b^*$ as in above Propositions. Then

1. *If $w > w_f^*$, there exists a unique equilibrium such that $G_d^{-t} P_t$ converges to a positive number, which is fundamental.*
2. *If $w < w_b^*$, there exists an equilibrium such that $G^{-t} P_t$ converges to a positive number, which is bubbly. If in addition $w < w_f^*$, there exist no fundamental equilibria.*
3. *If $w_f^* < w < w_b^*$, there exist a continuum of equilibria such that $G^{-t} P_t$ converges to zero, which are all bubbly except the unique equilibrium in 1.*

Necessity of stock and land bubbles

- Finally, we present simple model to show necessity of stock and land bubbles
- Two-sector two-period OLG model
  - Log utility $(1-\beta)\log y_t + \beta \log z_{t+1}$
  - capital-intensive sector: production function $F(K_t, L_t)$, with exogenous capital K_t and labor L_t (for simplicity but inessential)
  - land-intensive sector: output $D_t X$, where X: land supply, D_t: dividend (e.g., fruit of Lucas tree)
- N, X: number of stocks (claim to capital) and land supply
- Q_t, P_t: stock and land price

Equilibrium

- Firms maximize profit $F(K, L) - r_t K - w_t L$
- By first-order condition, rent $r_t = F_K(K_t, L_t)$, wage $w_t = F_L(K_t, L_t)$
- No-arbitrage condition

$$R_t := \frac{Q_{t+1} + r_{t+1}K_{t+1}/N}{Q_t} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

- Define
  - Aggregate asset value $S_t := Q_t N + P_t X$
  - Aggregate dividend $E_t := r_t K_t + D_t X$
- Easy to show

$$R_t = \frac{S_{t+1} + E_{t+1}}{S_t}$$

Equilibrium

- Due to log utility, aggregate savings $S_t = \beta w_t L_t = \beta F_L(K_t, L_t) L_t$

Proposition

In equilibrium, the aggregate asset value, aggregate dividend, and gross risk-free rate are uniquely given by

$$S_t = \beta F_L(K_t, L_t) L_t,$$
$$E_t = F_K(K_t, L_t) K_t + D_t X,$$
$$R_t = \frac{S_{t+1} + E_{t+1}}{S_t}.$$

There is a bubble in the aggregate asset market if and only if

$$\sum_{t=1}^{\infty} \frac{F_K(K_t, L_t) K_t + D_t X}{F_L(K_t, L_t) L_t} < \infty.$$

Bubble substitution

- Clearly aggregate asset value S_t uniquely determined
- But stock and land prices indeterminate, because we can assign bubble arbitrarily:
  - Let $V_t = V_t^S N + V_t^L X \leq S_t$ be fundamental value of aggregate asset
  - Let $B_t = S_t - V_t \geq 0$ aggregate bubble
  - Take any $\theta \in [0, 1]$, and define stock and land price by

$$
\begin{aligned}
Q_t &= V_t^S + \frac{\theta}{N} B_t, \\
P_t &= V_t^L + \frac{1-\theta}{X} B_t
\end{aligned}
$$

- This indeterminacy unimportant, because macroeconomic implications identical across equilibria

Productivity growth and bubbles

- As illustration, consider following example
- Production function is CES

$$F(K, L) = \left(\alpha K^{1-1/\sigma} + (1-\alpha)L^{1-1/\sigma}\right)^{\frac{1}{1-1/\sigma}},$$

with elasticity of substitution $\sigma < 1$ (empirically relevant case)

- $(K_t, L_t, D_t) = (K_0 G_K^t, L_0 G_L^t, D_0 G_X^t)$

Productivity growth and bubbles

- Can show

$$F_K(K,L) = \left(\alpha K^{1-1/\sigma} + (1-\alpha)L^{1-1/\sigma}\right)^{\frac{1}{\sigma-1}} \alpha K^{-1/\sigma},$$

$$F_L(K,L) = \left(\alpha K^{1-1/\sigma} + (1-\alpha)L^{1-1/\sigma}\right)^{\frac{1}{\sigma-1}} (1-\alpha)L^{-1/\sigma}.$$

- Using this, we can prove:

Proposition

If the production function is CES with $\sigma < 1$ and

$$(K_t, L_t, D_t) = (K_0 G_K^t, L_0 G_L^t, D_0 G_X^t),$$

then there is a bubble in the aggregate asset market if and only if $G_K > G_L > G_X$.

Concluding remarks

- Modern macro-finance theory has generally considered rational bubbles fragile
- But this is because many models are preoccupied with balanced growth
- Once we consider unbalanced growth (e.g., different technological progress in different sectors), asset price bubbles naturally and necessarily emerge

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