A Theory of the Saving Rate of the Rich

Qingyin Ma[1] Alexis Akira Toda[2]

[1]Capital University of Economics and Business

[2]UCSD

May 13, 2020

Empirically, the rich save more

- Fagereng, Holm, Moll, & Natvik (2019) ▸ Model

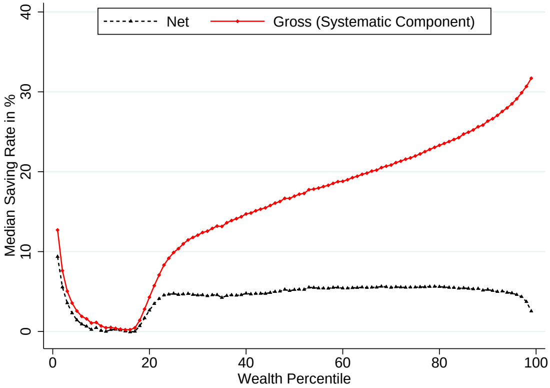


Figure 1: Saving rates across the wealth distribution (preview).

Empirically, the rich save more

- Understanding saving behavior of the rich is important because
  - If rich have lower marginal propensity to consume (MPC), then consumption tax regressive and may not be desirable from equity perspectives
  - MPC heterogeneity implies wealth distribution matters for determining aggregate consumption and hence for monetary policy (Kaplan *et al.*, 2018)

Homotheticity v.s. non-homotheticity

- High saving rate of rich seem to contradict homotheticity
  - Homothetic preferences $\implies$ (asymptotically) linear policies
  - Hence asymptotically constant saving rate
- Most explanations of high saving rate of rich based on non-homothetic preferences
  - Carroll (2000): 'capitalist spirit' (utility from holding wealth)
  - De Nardi (2004): bequest

Homotheticity v.s. non-homotheticity

- High saving rate of rich seem to contradict homotheticity
  - Homothetic preferences $\implies$ (asymptotically) linear policies
  - Hence asymptotically constant saving rate
- Most explanations of high saving rate of rich based on non-homothetic preferences
  - Carroll (2000): 'capitalist spirit' (utility from holding wealth)
  - De Nardi (2004): bequest
- However, non-homothetic preferences have undesirable properties
  - Inconsistent with balanced growth
  - Many parameters and calibration arbitrary

Contributions

1. "Homothetic theory" of high saving rate of the rich
   - (Technical) Prove asymptotic linearity of consumption functions
   $$\lim_{a \to \infty} \frac{c(a, z)}{a} = \bar{c}(z) = \text{constant}$$
   in general Markovian setting with stochastic discount factor β, returns R, and income Y
   - (Technical) Exact analytical characterization of asymptotic MPC $\bar{c}(z)$
   - (Surprising) Necessary and sufficient condition for $\bar{c}(z) = 0$
2. Calibrate model and show zero asymptotic MPC (hence increasing and large saving rate) empirically plausible

Literature

- **Income fluctuation problem:** Schechtman & Escudero (1977 JET); Chamberlain & Wilson (2000 RED); Li & Stachurski (2014 JEDC); Ma, Stachurski, & Toda (2020 JET)
- **Concavity of consumption:** Carroll & Kimball (1996 ECMA)
- **Saving rate:** Dynan, Skinner, & Zeldes (2004 JPE); Fagereng, Holm, Moll, & Natvik (2019 WP)
- **Asymptotic linearity (heuristic):** Toda (2019 JME); Gouin-Bonenfant & Toda (2018 WP)
- **Other properties:** Carroll (2009 JME); Carroll (2020 QE)

Income fluctuation problem

Consider

$$\begin{aligned} &\text{maximize} && \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\text{subject to} && a_{t+1} = R_{t+1}(a_t - c_t) + Y_{t+1}, \\ &&& 0 \le c_t \le a_t, \end{aligned}$$

where

- β: discount factor
- c_t, Y_t: consumption and non-financial income
- a_t: asset at beginning of time t including current income
- R_t: asset return from $t-1$ to t

(More general) income fluctuation problem

Consider

$$\begin{aligned} \text{maximize} \quad & \mathrm{E}_0 \sum_{t=0}^{\infty} \left(\prod_{i=0}^{t} \beta_i \right) u(c_t) \\ \text{subject to} \quad & a_{t+1} = R_{t+1}(a_t - c_t) + Y_{t+1}, \\ & 0 \leq c_t \leq a_t, \end{aligned}$$

where

- β_t: discount factor from time $t-1$ to t (set $\beta_0 \equiv 1$)
- c_t, Y_t: consumption and non-financial income
- a_t: asset at beginning of time t including current income
- R_t: asset return from $t-1$ to t

Stochastic processes

Stochastic processes $\{\beta_t, R_t, Y_t\}_{t \geq 1}$ obey

$$\beta_t = \beta(Z_t, \varepsilon_t), \quad R_t = R(Z_t, \zeta_t), \quad Y_t = Y(Z_t, \eta_t),$$

where

- β, R, Y: nonnegative measurable functions
- $\{Z_t\}_{t \geq 0}$: time-homogeneous finite state Markov chain taking values in $\mathsf{Z} = \{1, \dots, Z\}$ with transition probability matrix P
- innovation processes $\{\varepsilon_t\}, \{\zeta_t\}, \{\eta_t\}$ IID over time and mutually independent

Assumptions

A1 (Inada condition)

$u : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ is twice continuously differentiable on $(0, \infty)$ and satisfies $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$

A2 (spectral condition)

The following conditions hold:

1. $\mathrm{E}_z \, \beta < \infty$ and $\mathrm{E}_z \, \beta R < \infty$ for all $z \in \mathsf{Z}$
2. $r(PD_\beta) < 1$ and $r(PD_{\beta R}) < 1$
3. $\mathrm{E}_z \, Y < \infty$ and $\mathrm{E}_z \, u'(Y) < \infty$ for all $z \in \mathsf{Z}$

Here

- r: spectral radius
- D_X: diagonal matrix with $D_X(z, z) = \mathrm{E}_z \, X = \mathrm{E}\,[X \mid Z = z]$

Existence and uniqueness

Theorem (Ma, Stachurski, Toda (2020), Theorem 2.2)

Suppose A1–A2 hold. Then the income fluctuation problem has a unique solution. Furthermore, the consumption function $c(a, z)$ *can be computed by policy function iteration.*

Existence and uniqueness

Theorem (Ma, Stachurski, Toda (2020), Theorem 2.2)

Suppose A1–A2 hold. Then the income fluctuation problem has a unique solution. Furthermore, the consumption function $c(a, z)$ can be computed by policy function iteration.

- Because borrowing constraint $c_t \leq a_t$ may bind, Euler equation becomes

$$u'(c_t) = \max\left\{\mathrm{E}_t\, \beta_{t+1} R_{t+1} u'(c_{t+1}), u'(a_t)\right\}$$

- Given candidate policy $c(a, z)$, policy function iteration updates $c(a, z)$ by $\xi = Tc(a, z)$, where

$$u'(\xi) = \max\left\{\mathrm{E}_z\, \hat{\beta} \hat{R} u'(c(\hat{R}(a - \xi) + \hat{Y}, \hat{Z})), u'(a)\right\}$$

Additional assumptions for asymptotic linearity

A1' (CRRA)

The utility function exhibits constant relative risk aversion $\gamma > 0$:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & (\gamma \neq 1) \\ \log c. & (\gamma = 1) \end{cases}$$

Furthermore, $\mathrm{E}_z \beta R^{1-\gamma} < \infty$ for all z.

- Condition $\mathrm{E}_z \beta R^{1-\gamma} < \infty$ unnecessary but makes exposition simpler

Heuristic derivation of asymptotic MPC

- u is CRRA, so $u'(c) = c^{-\gamma}$
- Setting $c(a,z) \approx \bar{c}(z)a$ (linear), Euler equation becomes

$$\bar{c}(z)^{-\gamma} \approx \mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}\bar{c}(\hat{Z})^{-\gamma}(1-\bar{c}(z))^{-\gamma}$$

- Setting $x(z) = \bar{c}(z)^{-\gamma}$, we get

$$x(z) \approx \left(1 + \left(\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x(\hat{Z})\right)^{1/\gamma}\right)^{\gamma}$$

- Setting $D = D_{\beta R^{1-\gamma}}$, we get

$$x(z) \approx (Fx)(z) := \left(1 + (PDx)(z)^{1/\gamma}\right)^{\gamma},$$

so x should be fixed point of F

Theorem (Asymptotic linearity)

Suppose A1' and A2 hold, $c(a,z)$ be consumption function, and $D = D_{\beta R^{1-\gamma}}$.

1. *If $r(PD) < 1$, then*

$$\bar{c}(z) := \lim_{a \to \infty} \frac{c(a,z)}{a} = x^*(z)^{-1/\gamma}$$

for all $z \in Z$, where $x^* = (x^*(z))_{z=1}^{Z} \in \mathbb{R}_+^Z$ is unique fixed point of $F : \mathbb{R}_+^Z \to \mathbb{R}_+^Z$ defined by

$$(Fx)(z) := \left(1 + (PDx)(z)^{1/\gamma}\right)^{\gamma}$$

2. *If $r(PD) \geq 1$ and PD irreducible, then $\lim_{a \to \infty} c(a,z)/a = 0$*

Discussion

- In typical income fluctuation problem, people assume "finite value condition" $\mathrm{E}_z\,\beta R^{1-\gamma} < 1$, but unnecessary
  - p. 244 of Samuelson (1969 REStat), Eq. (9) of Krebs (2006 ET), Eq. (3) of Carroll (2009 JME), Eq. (18) of Toda (2014 JET), Eq. (3) of Toda (2019 JME)
- When $\mathrm{E}_z\,\beta R^{1-\gamma} \geq 1$, asymptotic MPC can be zero (surprising)
- Theorem does not cover all cases because assumes $\mathrm{E}_z\,\beta R^{1-\gamma} < \infty$ and requires irreducibility of PD, but these assumptions can be dropped

General case

- Let $K = PD$, where P: transition probability matrix, D: diagonal with $D(z,z) = \mathrm{E}_z\, \beta R^{1-\gamma} \in [0, \infty]$
  - Use convention $\beta R^{1-\gamma} = (\beta R) R^{-\gamma}$ and $0 \cdot \infty = 0$, so always well-defined
- Relabel states such that

$$K = \begin{bmatrix} K_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_J \end{bmatrix},$$

where each diagonal block K_j irreducible
  - Recall: square matrix A reducible if $\exists$ permutation matrix P such that $P^\top AP$ is block upper triangular with at least two diagonal blocks
  - Hence irreducible decomposition of K always exists

Complete characterization

Theorem

Suppose A2 holds and utility is CRRA (γ). Express $K = PD$ as block upper triangular with irreducible diagonal blocks. Define $\{x_n\}_{n=0}^{\infty} \in [0,\infty]^Z$ by $x_0 = 1$ and $x_n = Fx_{n-1}$, where F is as before. Then $\{x_n\}$ monotonically converges to $x^* \in [1,\infty]^Z$ and

$$\bar{c}(z) := \lim_{a \to \infty} \frac{c(a,z)}{a} = x^*(z)^{-1/\gamma} \in [0,1].$$

Furthermore, $\bar{c}(z) = 0$ if and only if there exist j, $\hat{z} \in Z_j$, and $m \in \mathbb{N}$ such that $K^m(z,\hat{z}) > 0$ and $r(K_j) \geq 1$.

Example: log utility

- If $\gamma = 1$, then $x^* = Fx^*$ becomes

$$x^* = 1 + PDx^* \iff x^* = (I - PD)^{-1}1,$$

where $D = D_\beta = \operatorname{diag}(\dots, \mathrm{E}_z\, \beta, \dots)$

- Since $r(PD) < 1$ by A2, we always have $\bar{c}(z) > 0$

Example: IID returns

- If $b = b(z) = \mathrm{E}_z\, \beta R^{1-\gamma}$ does not depend on z, then $D = bI$
- If $x = k1$ is a multiple of the vector 1, then $PDx = bPk1 = bk1$ because P is transition probability matrix
- Hence if $b < 1$, $x^* = Fx^*$ reduces to

$$x^*(z) = (1 + (bx^*(z))^{1/\gamma})^\gamma \iff \bar{c}(z) = x^*(z)^{-1/\gamma} = 1 - b^{1/\gamma}$$

- Therefore with constant discounting ($\beta(z, \varepsilon) \equiv \beta$) and risk-free saving ($R(z, \zeta) \equiv R$), asymptotic MPC is constant regardless of income shocks:

$$\bar{c}(z) = \begin{cases} 1 - (\beta R^{1-\gamma})^{1/\gamma} & \text{if } \beta R^{1-\gamma} < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Saving rates of the rich

- From budget constraint, saving rate (excluding capital loss) is

$$s_{t+1} = \frac{\overbrace{a_{t+1} - a_t}^{\text{change in wealth}}}{\underbrace{\max\{(R_{t+1} - 1)(a_t - c_t), 0\}}_{\text{capital gains}} + \underbrace{Y_{t+1}}_{\text{labor income}}}$$

$$= 1 - \frac{(\hat{R} - 1)^-(1 - c/a) + c/a}{(\hat{R} - 1)^+(1 - c/a) + \hat{Y}/a} \in (-\infty, 1)$$

- Letting $a \to \infty$, asymptotic saving rate is

$$\bar{s} := 1 - \frac{(\hat{R} - 1)^-(1 - \bar{c}) + \bar{c}}{(\hat{R} - 1)^+(1 - \bar{c})} \in [-\infty, 1]$$

Impossibility of positive saving rates

Proposition

Consider a canonical Bewley model in which agents are infinitely-lived and relative risk aversion γ, discount factor β, and return on wealth $R > 1$ are constant. Then in the stationary equilibrium the asymptotic saving rate is negative.

Proof.

- Stachurski & Toda (2019 JET) show $\beta R < 1$ in stationary equilibrium
- Since $R > 1$, we obtain $\beta R^{1-\gamma} = (\beta R) R^{-\gamma} < 1$. By previous example, asymptotic MPC is $\bar{c} = 1 - (\beta R^{1-\gamma})^{1/\gamma} \in (0, 1)$.
- Hence

$$\bar{s} = 1 - \frac{\bar{c}}{(R-1)(1-\bar{c})} < 0$$
$$\iff (R-1)(1-\bar{c}) < \bar{c}$$
$$\iff (R-1)(\beta R^{1-\gamma})^{1/\gamma} < 1 - (\beta R^{1-\gamma})^{1/\gamma}$$
$$\iff (\beta R)^{1/\gamma} < 1. \qquad \square$$

Stochastic β, R need not help

Proposition

Consider a Bewley model in which agents are infinitely-lived, relative risk aversion γ *is constant, and* $\{\beta_t, R_t\}_{t\geq 1}$ *is* IID *with* $\mathrm{E}\,R > 1$ *and* $\mathrm{E}\,\beta R^{1-\gamma} < 1$*. If the stationary equilibrium wealth distribution has an unbounded support, then the asymptotic saving rate evaluated at* $\hat{R} = \mathrm{E}\,R$ *is nonpositive.*

Proof.

- Since by assumption $\mathrm{E}\,\beta R^{1-\gamma} < 1$, by previous example the asymptotic MPC is $\bar{c} = 1 - (\mathrm{E}\,\beta R^{1-\gamma})^{1/\gamma} \in (0, 1)$.
- Hence asymptotic saving rate evaluated at $\mathrm{E}\,R > 1$ is

$$\begin{aligned} \bar{s} = 1 - \frac{\bar{c}}{(\mathrm{E}\,R - 1)(1 - \bar{c})} &\leq 0 \\ \iff (\mathrm{E}\,R - 1)(1 - \bar{c}) &\leq \bar{c} \\ \iff \mathrm{E}\,R(1 - \bar{c}) &\leq 1. \end{aligned}$$

- Since $\mathrm{E}\,R(1 - \bar{c})$ is the expected growth rate of wealth for infinitely wealthy agents, if wealth distribution unbounded and $\mathrm{E}\,R(1 - \bar{c}) > 1$, then wealth grow at the top, violating stationarity. □

Numerical example with $\bar{c} = 0$

- Constant discount factor β and RRA γ
- Gross portfolio return is

$$R_t(\theta) := 1 + (1-\tau)(\theta^s R_t^s + \theta^b R_t^b + \theta^f R^f - 1),$$

where R_t^s: stock return, R_t^b: business return, R^f: risk-free rate, τ: capital income tax

- Business return

$$R_t^b = \begin{cases} \frac{1}{1-p_b} R_t^s & \text{with probability } 1-p_b, \\ 0 & \text{with probability } p_b, \end{cases}$$

- Income growth deterministic: $Y_{t+1}/Y_t = e^g$

Calibration

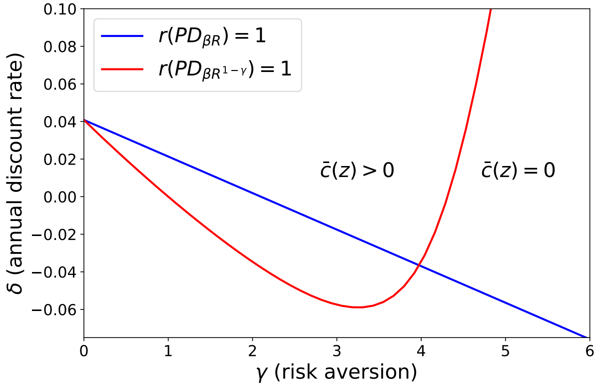
- One period is a month, annual 4% discounting
- Stock return GARCH(1, 1),

$$\log R_t^s = \mu - \frac{1}{2}\sigma_t^2 + \epsilon_t,$$
$$\epsilon_t = \sigma_t \zeta_t, \quad \zeta_t \sim \text{IID} N(0, 1)$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \rho \sigma_{t-1}^2,$$

  calibrated from monthly stock return and discretize using Farmer & Toda (2017)
- Business bankruptcy rate 2.5% following Luttmer (2010)
- Portfolio data constructed from Saez & Zucman (2016), $(\theta^s, \theta^b, \theta^f) = (0.5546, 0.0827, 0.3627)$
- Income growth g calibrated from real per capita GDP growth

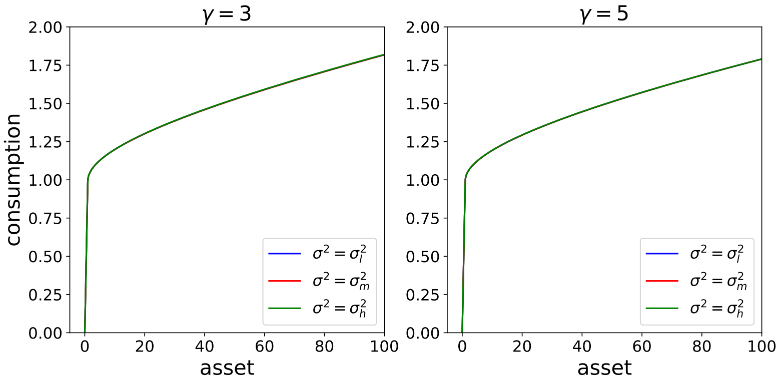
Asymptotic MPC with GARCH$(1, 1)$ returns

- Zero asymptotic MPC possible with γ above 4–5



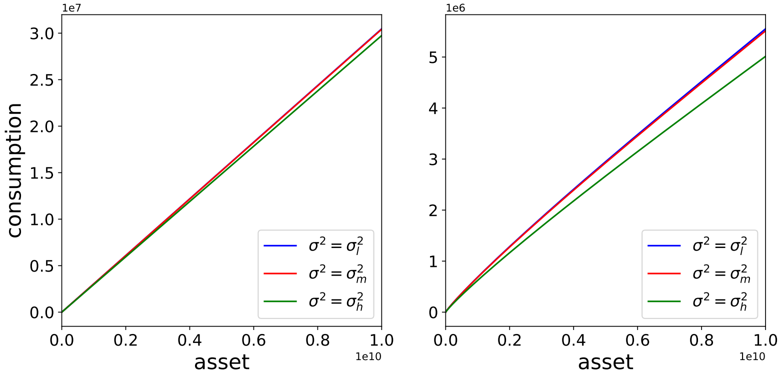
Consumption functions at low asset level

- Can't see any meaningful difference between $\gamma = 3, 5$



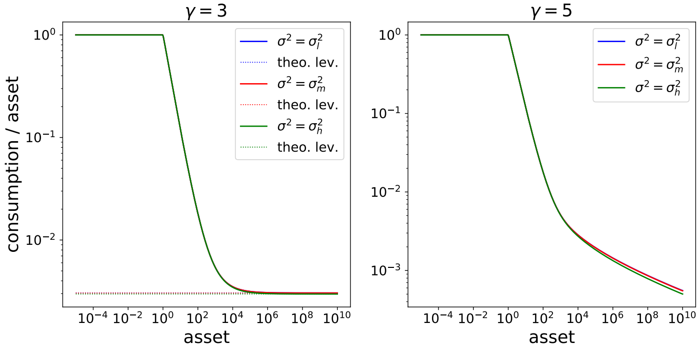
Consumption functions at high asset level

- Consumption with $\gamma = 5$ much lower and more concave



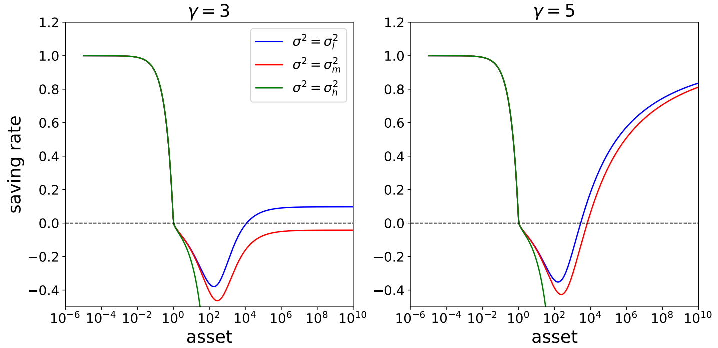
Consumption rate

- $\gamma = 3$: $r(PD) < 1$ and $\bar{c} > 0$
- $\gamma = 5$: $r(PD) \geq 1$ and $\bar{c} = 0$



Saving rate

- $\gamma = 3$: $r(PD) < 1$ and $\bar{s}$ small
- $\gamma = 5$: $r(PD) \geq 1$ and $\bar{s} = 1$ ▸ Data



Outline of proof

Show that

1. policy function iteration leads to increasingly tighter upper bounds on consumption functions that are asymptotically linear with explicit slopes,
2. slopes of upper bounds converge using fixed point theory of monotone convex maps, and
3. consumption functions have linear lower bounds with identical slopes to limit of upper bounds, implying asymptotic linearity.

Space of candidate consumption functions

- Let $\mathcal{C}$ be space of candidate consumption functions such that $c : (0, \infty) \times \mathrm{Z} \to \mathbb{R}$ is (i) continuous, (ii) increasing in first element, (iii) $0 < c(a, z) \leq a$ for all a, z, and (iv)

$$\sup_{(a,z)\in(0,\infty)\times \mathrm{Z}} \left| u'(c(a, z)) - u'(a) \right| < \infty$$

- For $c, d \in \mathcal{C}$, define marginal utility distance

$$\rho(c, d) = \sup_{(a,z)\in(0,\infty)\times \mathrm{Z}} \left| u'(c(a, z)) - u'(d(a, z)) \right| < \infty$$

- Ma, Stachurski, & Toda (2020) show $(\mathcal{C}, \rho)$ is complete metric space

Time iteration operator

- Given candidate policy $c \in \mathcal{C}$, define $Tc(a,z)$ by the value $\xi \in (0,a]$ that solves Euler equation

$$u'(\xi) = \max\left\{\mathrm{E}_z\,\hat{\beta}\hat{R}u'(c(\hat{R}(a-\xi)+\hat{Y},\hat{Z})),u'(a)\right\}$$

- Ma, Stachurski, & Toda (2020 JET) show $T : \mathcal{C} \to \mathcal{C}$ is contraction mapping

Iterating T leads to tighter upper bounds

Proposition

Let everything be as in Theorem. If $c \in \mathcal{C}$ and

$$\limsup_{a \to \infty} \frac{c(a,z)}{a} \leq x(z)^{-1/\gamma}$$

for some $x(z) \geq 1$ for all $z \in \mathsf{Z}$, then

$$\limsup_{a \to \infty} \frac{Tc(a,z)}{a} \leq (Fx)(z)^{-1/\gamma}.$$

Proof.

- Let $\{a_n, \alpha_n\}$ be sequence such that $a_n \uparrow \infty$ and $\alpha_n(z) = Tc(a_n, z)/a_n \to \limsup_{a\to\infty} Tc(a, z)/a$
- Use Euler equation, definition of T, and Fatou's lemma to show claim

□

Characterizing limit of iteration of F

Proposition

Let $(Fx)(z) := \left(1 + (PDx)(z)^{1/\gamma}\right)^{\gamma}$*. Then F has a (necessarily unique) fixed point* $x^* \in \mathbb{R}_+^Z$ *if and only if* $r(PD) < 1$*.*
Take any $x_0 \in \mathbb{R}_+^Z$ *and define* $x_n = Fx_{n-1}$ *for all* $n \in \mathbb{N}$*.*

1. *If* $r(PD) < 1$*, then* $x_n \to x^*$
2. *If* $r(PD) \geq 1$ *and PD irreducible, then* $x_n(z) \to \infty$

Proof.

- $F = \phi \circ K$, where $\phi(t) = (1 + t^{1/\gamma})^\gamma$ and $K = PD$
- ϕ increasing and concave (convex) if $\gamma \leq 1$ (> 1)
- Case $r(PD) < 1$: apply Du (1990) below to F
- If $\exists$ fixed point x^*, then $x^* = Fx^* \gg Kx^*$; multiplying left Perron vector y of K, get $y'x^* > r(K)y'x^*$, hence $r(K) < 1$ □

Theorem (Du, 1990)

If X partially ordered Banach space, $A : X \to X$ monotone, convex or concave, and $\exists u \leq v$ such that $Au \gg u$ and $Av \ll v$, then A has unique fixed point on $[u, v]$ and can be computed by iterating $x_n = Ax_{n-1}$

Lower bound

Proposition

Let everything be as in Theorem. Suppose $r(PD) < 1$ and let $x^* \in \mathbb{R}^Z_{++}$ unique fixed point of F. Restrict candidate space to

$$\mathcal{C}_0 = \{c \in \mathcal{C} \mid c(a, z) \geq \epsilon(z)a \quad \text{for all } a > 0 \text{ and } z \in Z\},$$

where $\epsilon(z) = x^*(z)^{-1/\gamma} \in (0, 1]$. Then $T\mathcal{C}_0 \subset \mathcal{C}_0$.

Corollary

Consumption function satisfies $c(a, z) \geq x^*(z)^{-1/\gamma} a$.

Proof.

Let $c_0(a, z) = a \in \mathcal{C}_0$. Iterating $T : \mathcal{C}_0 \to \mathcal{C}_0$, consumption function (fixed point of T) must be in $\mathcal{C}_0$. □

Proof of Proposition.

- If $T\mathcal{C}_0 \not\subset \mathcal{C}_0$, then $\exists c \in \mathcal{C}_0, a > 0, z \in \mathrm{Z}$ such that $\xi := Tc(a,z) < \epsilon(z)a \leq a$
- Using Euler equation and concavity of u (u' decreasing),

$$u'(\epsilon(z)a) < u'(\xi) = \mathrm{E}_z\, \hat{\beta}\hat{R}u'(c(\hat{R}(a-\xi)+\hat{Y},\hat{Z}))$$
$$\leq \mathrm{E}_z\, \hat{\beta}\hat{R}u'(\epsilon(\hat{Z})(\hat{R}(a-\xi)+\hat{Y})) \leq \mathrm{E}_z\, \hat{\beta}\hat{R}u'(\epsilon(\hat{Z})\hat{R}[1-\epsilon(z)]a)$$

- Using $u'(c) = c^{-\gamma}$ and $\epsilon(z) = x^*(z)^{-1/\gamma}$, we obtain

$$x^*(z) < \mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x^*(\hat{Z})[1-x^*(z)^{-1/\gamma}]^{-\gamma}$$
$$\iff x^*(z) < \left(1 + (\mathrm{E}_z\, \hat{\beta}\hat{R}^{1-\gamma}x^*(\hat{Z}))^{1/\gamma}\right)^{\gamma} = (Fx^*)(z),$$

contradiction because x^* fixed point of F □

Proof of Theorem: case $r(PD) \geq 1$

- Define $c_0 \in \mathcal{C}$ by $c_0(a, z) = a$ and $c_n := T^n c_0 \in \mathcal{C}$
- By previous result, $\limsup_{a\to\infty} c_n(a, z)/a \leq x_n(z)^{-1/\gamma}$, where $x_0 = 1$ and $x_n = Fx_{n-1}$
- Clearly $c(a, z) \leq a = c_0(a, z)$, so $c(a, z) \leq c_n(a, z)$ by induction
- If $r(PD) \geq 1$ and PD irreducible, then $x_n(z) \to \infty$, so

$$0 \leq \limsup_{a\to\infty} \frac{c(a, z)}{a} \leq \lim_{a\to\infty} \frac{c_n(a, z)}{a} = x_n(z)^{-1/\gamma} \to 0$$

as $n \to \infty$

Proof of Theorem: case $r(PD) < 1$

- By same argument,

$$\limsup_{a\to\infty} \frac{c(a,z)}{a} \leq \limsup_{a\to\infty} \frac{c_n(a,z)}{a} \leq x_n(z)^{-1/\gamma} \to x^*(z)^{-1/\gamma}$$

- But we know $c(a,z)/a \geq x^*(z)^{-1/\gamma}$
- Hence

$$\lim_{a\to\infty} \frac{c(a,z)}{a} = x^*(z)^{-1/\gamma}$$

Conclusion

- With homothetic preferences, policy functions are asymptotically linear
- Asymptotic linearity is expected but proof not simple
- Surprisingly, $\bar{c}(z) = \lim_{a\to\infty} c(a,z)/a = 0$ is possible
- May explain why the rich save so much